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Mathematical Models in Economics: Applications of Sequences, Derivatives, and Differential Equations

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Authors' contributions

This work was carried out in collaboration among all authors. All authors read and approved the final manuscript.

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ABSTRACT

This paper examines the application of mathematical tools—numerical sequences, derivatives, and differential equations—that have important uses within economic analyses on population modeling, investment growth, and population cost. Numerical examples shall be used to illustrate how such a mathematical approach helps in financial decision-making and long-term forecasting. It's expected that this work will provide insight into the benefits of mathematical models in addressing complex economic challenges and make a case for interdisciplinary approaches in economics.

Keywords: Sequences; derivatives; differential equations; marginal cost; average cost; GDP.

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1. INTRODUCTION

This manuscript is important to the scientific community as it reaches back into the very fundamental mathematics underlying economics, an area so key for policy analysis, financial forecasting, and strategic economic planning. By linking the mathematical models with real-world economic scenarios, like those that are concerned with population growth, as well as the calculation of interest, a foundation for making more robust, data-driven decisions is laid. Another strength of the paper is its focus on mathematical applications that help reach an interdisciplinary understanding, showing economists and mathematicians that their work has relevance to problems in economics.

In other words, the study fosters collaboration between mathematics and economics**,** highlights how mathematical modelling aids in financial decision-making and long-term forecasting and underlines the importance of quantitative methods in addressing complex problems in economics.

1.1 Application of Sequences in Economy

Population changes depend upon such factors as nutrition, natural catastrophes, wars, economic wealth, cultural perceptions of family, and especially on time. Over time, the population may remain constant, increase, or decrease.

This process can be formulated using mathematical terms, with the use of some Math functions and how to visualize it with the connection between Math, Economy, and Informatics or programming (Sigal, 2017). The number of population in a certain year is a natural number, so if we want to study the development during in period of time, then the numbers of population in this period of time are a subset of natural numbers (Falahati, 2019, Lubis et al.,2024).

Suppose in this year we have N_0 inhabitants in the Republic of North Macedonia, then after a year we have N_1 inhabitants, after two years N_2, and after i years we have N_i inhabitants. So we can generate the finite sequence

$$N_1, N_2, \dots, N_i$$

where N_k tell the number of inhabitants after k-years, $k = 1, 2, 3, \dots i$

Below, we present the population in the Republic of North Macedonia, over the years:

Table 1. Population at the end of the certian year, in North Macedonia

	2009	**2014**	**2019**
Population at the end of the year	2052722	2069172	2076255

From the Table 1, we can construct a sequence, in this way:

$$N_{2009} = 2052722, \quad N_{2014} = 2069172, \quad N_{2019} = 2076255$$

when N_i denoted the number of the population at the end of the year i. So, based on this finite sequence we have an increased sequence, which means the population in North Macedonia from 2009 to 2019 is increasing continuously.

Now, we can give the formal definition of finite sequence:

Definition 1 A sequence is a function whose domain is the positive integers.

The following are examples of sequences:

i. $\mathrm{f(n)} = 2\mathrm{n}$ or $2, 4, 6, 8, 10, \dots$

ii. $\mathrm{f(n)} = \frac{1}{\mathrm{n}}$ or $1, 1/2, 1/3, 1/4, \dots$

iii. $\mathrm{f(n)} = (-1)^{\mathrm{n}}$ or $-1, 1, -1, 1, \dots$

The reason for giving the population was because it has an important impact on the economy of a state and the direction of the economics and politics of a government. So if we study like sequences we can predict consequences and the state's institutions could take adequate action (Tarasov, 2019, Ramadani et al., 2024).

Another well-known sequence in mathematics, which is used in economics, especially in finance is the sequence, when Jacob Bernoulli, in 1683 was studying compound interest, especially continuous compound interest.

Suppose you lend money to your friend at a 100% interest rate, compounded continuously, year by year. In the next year, your money would double. If the interest rate would be 50% at every six months, then your money would grow 225% in one year.

So, as the interval gets smaller, the total return gets slightly higher. If the interest rate calculated n times per year with a rate $100\%/n$, then the total wealth at the end of the first year will growth 2.7 times, compared with the initial wealth, when n is sufficiently large.

Let's take an example. Suppose you have $20000€$ paying 6% interest with continuous compounding, after ten years, how many euros do you have? https://mathshistory.st-andrews.ac.uk/HistTopics/e/

This financial problem is solved by the following formula

$$FV = PVe^{rt} \ldots \quad (1),$$

Where

$FV = \ future\ value$

$PV = \ present\ value\ of\ balance\ or\ sum$

$e = Euler's\ number, (e = 2.71828 \ldots)$

$r = interest\ rate\ being\ compounded$

$t = time\ in\ years$

So, by the problem, we have that $PV = 20000€$**,** $r = 0.06$, $t = 10$, then

$$FV = 20000 \cdot 2.71828^{0.06 \cdot 10} = 36442.3613€ \ldots (2)$$

This amount of interest can calculated in another way, with the formula

$$FV = PV \left(1 + \frac{r}{n}\right)^{nt} \ldots \quad (3)$$

where n is represented the number of compounding periods in a year, in our case 12, then

$$\begin{aligned} FV = 20000 \left(1 + \frac{0.06}{12}\right)^{12 \cdot 10} \\ = 20000(1 + 0.005)^{120} = \\ = 20000 \cdot 1.005^{120} = 36387.9346 \\ = 20000 \cdot 1.005^{120} \\ = 36387.93468 \end{aligned}$$

If we compare the two last results the difference is small, considering the number of years.

But, if we want to know how long we should wait to double our initial investment, so if we save $20000€$**,** fixed interest rate 6% paid annually?

Now, we have invested an amount P and interest is paid annually at interest rate i , $\mathrm{i} - \text{the interest rate}$, after one year we have $P(1 + i)$, after two years

$\big(P(1 + i)\big)(1 + i) = P(1 + i)^2$, and after t years $P(1 + i)^t$, so we can write the equation, by the condition of the problem:

$$20000(1 + 0.06)^t = 40000$$

$$1.06^t = 2$$

$$t \log 1.06 = \log 2$$

$$t = \frac{log2}{log1.06} = 11.8956610459$$

So we should wait 12 years to double our initial investment. (Mathematics for Economics, 2001, https://www.investopedia.com/)

2. APPLICATION OF DERIVATIVE IN ECONOMICS

In the 1970's in economic world happened a revolution, which is called marginal revolution. From production to consumption, from supply to demand, and from cost to utility were some of the features of this revolution. In economics terminology, it contains an analysis of the marginal utility theory of value and the extensive application of the marginal analysis method. In fact, the last method is based on the mathematical analysis method, to express more explicitly on concept of derivative.

The mathematical definition of the derivative of a function is defined by:

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x+x_0) - f(x_0)}{x - x_0} = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

where f is a function defined in a subset of real numbers and for its exists the above limit.

In economic terminology, the derivative of a function is called its marginal function and the value at a specific point is called the marginal value.

Marginal cost is related closely to cost minimization, in other words, is the change of total cost for each additional unit of goods that are produced if the other conditions remain the same.

If the product put [put is Q and its corresponding cost function is $\mathrm{C(Q)}$ is differentiable, hence the marginal cost is the derivative of the cost function with respect to output.

Now, let the problem of a factory of kinds of pasta be given as follows:

Suppose that their output is Q and the cost function

$$\mathrm{C(Q)} = 0{,}7\mathrm{Q}^2 + 234 \qquad (4),$$

then the marginal cost function will be $\mathrm{C}'(\mathrm{Q}) = 1.4\mathrm{Q}$. So if we take $\mathrm{C}'(5) = 1.4 \cdot 5$=7, $\mathrm{C}'(50) = 70$ and $\mathrm{C}'(500) = 700$, so the marginal cost of producing 5 pasta is 7, 50 is 70, and 500 is 700.

Now, let's see the average cost:

$$\frac{C(5)}{5} = 50.3 \ \ldots \qquad (5)$$

$$\frac{C(50)}{50} = 39.68 \ \ldots \qquad (6)$$

and

$$\frac{C(500)}{500} = 350.468 \ \ldots \qquad (7)$$

When $\mathrm{Q} = 5$ the output must be increased because the marginal cost is lower than the average cost, but in two other cases is not good to increase the output. (Phelps, 1961, 6. https://intro.quantecon.org/geom_series.html, Ramadani et al., 2024).

2.1 Application of Differential Equations in Economics

Ordinary differential equations have different applications, as in mathematics, and also in other fields outside of mathematics, in our case even economics, respectively in the analysis of market models.

Mathematically a linear differential equation is expressed with the formula:

$$\mathrm{a_n(x)}\frac{\mathrm{d^n y}}{\mathrm{dx^n}} + \mathrm{a_{n-1}(x)}\frac{\mathrm{d^{n-1}y}}{\mathrm{dx^{n-1}}} + \ldots + \mathrm{a_1(x)}\frac{\mathrm{dy}}{\mathrm{dx}} + \mathrm{a_0(x)y} = \mathrm{g(x)},$$

where

y is an unknown function, x is an independent variable a_i represents the coefficient functions and $g(x)$ is forced function.

2.2 To Calculate the Change in Gross Domestic Product with Time

If $\mathrm{Y(t)}$is the current state of GDP, then $\frac{dY}{dt}$ is the rate of change of the current state with respect to time and it is proportional to the current GDP, thus mathematically can we write

$$\frac{dY}{dt} = gY(t) \ \ldots \qquad (8)$$

g -is the growth rate.

2.3 To Explain an Economy's Growth Rate

These purposes economists have created different models, in which model they used differential equations, we will mention two of them.

2.3.1 Harrod-Domar model

This model was constructed by Roy. F. Harrod in 1939 and by Evsey Domar in 1946, in an independent manner. On based of this model, an economy's growth rate is explained in terms of saving and productivity of capital.

Its mathematical representation is

$$\frac{\dot{Y}}{Y} = sc - \delta \ \ldots \qquad (9)$$

Where $\frac{\dot{Y}}{Y}$ represents the output growth rate,

$\dot{\mathrm{Y}}$represents thederivative of Y,

c represents the marginal product of capital,

δ represents the rate of depreciation of capital stock, an

s represents the saving growth rate (Ordinary differential equation and its application, 2023).

2.3.2 Solow-swan model

Even this model was developed independently by Robert Solow and Trevor Swan in 1956. They try to explain the long-run economic growth, in terms of capital accumulation, population growth, and technological progress.

The equation which describes this model is

$$\mathrm{Y} = \mathrm{K}^{\alpha}(\mathrm{AL})^{1-\alpha}, 0 < \alpha < 1 \quad \dots \qquad (10)$$

Y-output, K and L labour and capital used, α elasticity of output with respect to capital, A labour augmenting technology, AL effective labour.

The marginal product of capital is given as

$$\mathrm{MP_k} = \frac{\partial \mathrm{Y}}{\partial \mathrm{K}} = \frac{\alpha \mathrm{A}^{1-\alpha}}{(\frac{\mathrm{K}}{\mathrm{L}})^{1-\alpha}} \quad \dots \qquad (11)$$

This is a single ordinary differential equation, which is non-linear, in pursuit of giving insight into long-run economic growth (Ordinary differential equation and its application, 2023, Falahati, 2019)

2.3.3 Samuelson

Paul Samuelson, in 1941, used in his paper differential equation in order to study the stability of equilibrium for several demand-supply scenarios

Let $\mathrm{D}(\mathrm{p}, \alpha)$ and $\mathrm{S}(\mathrm{p})$ be demand and supply functions price p , α shift parameter, which represents taste. At equilibrium, the price p^* and quantity q^*, are given by

$$q^* = \mathrm{D}(\mathrm{p}^*, \alpha) = \mathrm{S}(\mathrm{p}^*), \quad \dots \quad (12)$$

Where derivative of D with respect to α is greater than 0, but with respect to p is less than 0.

2.3.4 Phelps

He has developed a model based on the neoclassical growth model to study the consumption per unit of labor at equilibrium. The consumption per unit of labor is given by

$$\mathrm{c}(\mathrm{t}) = \mathrm{f}(\mathrm{k}) - \mathrm{nk} \quad \dots \qquad (13)$$

While for maximum consumption per unit of labor is given by:

$$\frac{dc}{dk} = \frac{df}{dk} - n = 0 \dots \qquad (14)$$

The second derivative of f with respect to k is less than zero, it means that the point of maximum is given by $\frac{df}{dk} = n$.

Therefore, we can conclude that marginal output for workers must equal the growth rate of the labour force (Ordinary differential equation and its application, 2023, Phelps, 1961).

2.4 Control of Production and Consumption

An enterprise generates revenue, where one part of it is distributed as a profit and the others saved for the other investiments. Suppose that the revenue is proportional with the part which are reinvested.

We define $\mathrm{y}(\mathrm{t})$ to be the output produced at a time $\mathrm{t} \geq 0$. Moreover, $\mathrm{u}(\mathrm{t})$ be the portion of the gain which is reinvested at the instance of time t. From these definitions, we can formulate the condition:
$0 \leq \mathrm{u}(\mathrm{t}) \leq 1$ for all $\mathrm{t} \geq 0$.

If a control u is given then the follow problem shuold be solved

$$\mathrm{y}'(\mathrm{t}) = \mathrm{ku}(\mathrm{t})\mathrm{y}(\mathrm{t})$$

$$\mathrm{y}(0) = 0 \ \dots \qquad (15)$$

where k -rate by which the invested capital grows.

For an period of time the quantity $(1 - \mathrm{u}(\mathrm{t}))\mathrm{y}(\mathrm{t})$ is the profit, which means it will not reinvest.
Therefore, we desire the maximization of the amount

$$\mathcal{F}(\mathrm{y}, \mathrm{u}) = \int_0^{\mathrm{T}} \big(1 - \mathrm{u}(\mathrm{t})\big)\mathrm{y}(\mathrm{t})\mathrm{dt} \quad \dots \qquad (16),$$

Where T –given instance of time (Eck, 2017).

Furthermore, differential equations have a wide range of applications in the economy: to study trade cycles, economic chaos, dynamic stability conditions of equilibrium et cetera.

3. APPLICATION OF STATISTICAL AND ZERO-SUM GAMES IN ECONOMICS

Let $\mathrm{H}_1(\mathrm{x}, \mathrm{y})$ and $\mathrm{H}_2(\mathrm{x}, \mathrm{y})$ be be arbitrary functions defined on the set product $\mathrm{X} \times \mathrm{Y}$. Exist a special case where $H_1(x,y) + H_2(x,y) = 0$ for all (x,y), … (17).

Otherwise in theory of game they are called payoffs and games zero-sum games or antagonistic games. In this kind of games, players have opposite goals- the payoff of a player equals the loss of the opponent. We have to find a the payoff function of player H in order to describe the game, completely. Each player choose his strategy, which are on the contrary with each-other. First player seek to maximize the function H, whereas the second tends to minimize it. Zero-sum games satisfy all the properties of normal-form games (Mazalov, 2014).

4. CONCLUSION

This concept is present in different fields, where we have to do “decision making”, in our case it is widespread in the economy because it is part of every day of businessmen, and scientists of the economy. It uses optimal allocation of resources in different assets. Some authors consider the application of a game-theoretical approach can impact in security risks.

Distribution of the economic environment allows the decision-maker to have greater flexibility when analyzing different economic systems.

The application of game theory in decision-making consists of the construction of a payoff matrix, which is in the most time-consuming phase. In the game-theoretic modeling in the economy, not all entries of the matrix are known, even some of them are unknown absolutely. This model helps us to estimate the economic environment for finding them.

Studying these unknown probability values is based on some type of ordering relation, whose studied by Fishburn and Trukhaev. In this direction have to mention Fishburn point estimates, Fishburn arithmetic progression, and Fishburn geometric progression, (Remesnik, 2014, Sigal, 2017).

DISCLAIMER (ARTIFICIAL INTELLIGENCE)

Author(s) hereby declare that NO generative AI technologies such as Large Language Models (ChatGPT, COPILOT, etc.) and text-to-image generators have been used during the writing or editing of this manuscript.

COMPETING INTERESTS

Authors have declared that no competing interests exist.

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