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Analytic solutions of time fractional diffusion equations by fractional reduced differential transform method (FRDTM)

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This paper examines a general recurrence relation by the use of fractional reduced differential transform and then a scheme (methodology) on how to find closed solutions of one dimensional time fractional diffusion equations with initial conditions in the form of infinite fractional power series and in terms of Mittag-Leffler function in one parameter as well as their exact solutions by the use of fractional reduced differential transform method. The new general recurrence relation and the methodology of the fractional reduced differential transform method were successfully developed. The obtained new general recurrence relation helps us to solve time-fractional diffusion equations with initial conditions and various external forces by using fractional reduced differential transform method. To see its effectiveness and applicability, five test examples were presented. The results show that the general recurrence relation works successfully in solving time-fractional diffusion equations in a direct way without using linearization, transformations, perturbation, discretization or restrictive assumptions by using fractional reduced differential transform method.

Key words: Time fractional diffusion equations with initial conditions, Caputo fractional derivatives, Mittag-Leffler function, Fractional reduced differential transform method (FRDTM).

INTRODUCTION

The beginning of fractional calculus is considered to be the L'Hopital's letter that raised the question: "What does $\frac{\partial^n f(x)}{\partial x^n}$ mean if $n=\frac{1}{2}$?" to Leibniz in 1695, where the notation for differentiation of non-integer order $\frac{1}{2}$ was discussed (Diethelm, 2010; Hilfer, 2000; Lazarevic et al., 2014; Millar and Ross, 1993; Ortigueira, 2011; Kumar and Saxena, 2016). Since then, many famous mathematicians such as Laplace, Fourier, Abel, Liouville, Riemann, Grunwald, Letnkov, Levy, Marchaud, Erdelyi and Reisz have worked much on this question and other related questions up to the middle of the 19th century, and provided important contributions to those creating the

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field which is known today as fractional calculus (Oldham and Spanier, 1974).

Fractional calculus theory is a mathematical analysis tool to the study of integrals and derivatives of arbitrary order, which unify and generalize the notations of integer-order differentiation and $n-$fold integration (El-Ajou et al., 2013; Millar and Ross, 1993; Oldham and Spanier, 1974; Podlubny, 1999). Although fractional calculus is almost as old as the classical calculus, it was only in recent few decades that its theory and applications have rapidly developed. It was Ross who organized the first international conference on fractional calculus and its applications at the University of new Haven in June 1974, and edited the proceedings (Ross, 1975). Oldham and Spanier (1974) published the first monograph on fractional calculus in 1974. Thereafter, because of the fact that fractional derivatives and integrals are non-local operators and then this property make them a powerful instrument for the description of memory and hereditary properties of different substances (Podlubny, 1999); theory and applications of fractional calculus have attracted much interest and become a pulsating research area.

Due to this, fractional calculus has got many important applications in numerous diverse and widespread areas of different fields of science, engineering and finance. For instance, in a book written by Shanantu Das (2011), it was discussed that fractional calculus is applicable to problems in: fractance circuits, electrochemistry, capacitor theory, feedback control system, vibration damping system, diffusion process, electrical science, and material creep. Fractional Calculus is applicable to problems in fitting experimental data, electric circuits, electro-analytical chemistry, fractional multi-poles, neurons and biology (Podlubny, 1999). It is also applicable to problems in polymer science, polymer physics, biophysics, rheology, and thermodynamics (Hilfer, 2000). In addition, it is applicable to problems in: electrochemical process (Millar and Ross, 1993; Oldham and Spanier, 1974; Podlubny, 1999), control theory (David et al., 2011; Podlubny, 1999), physics (Sabatier et al., 2007), science and engineering (Kumar and Saxena, 2016), transport in semi-infinite medium (Oldham and Spanier, 1974), signal processing (Sheng et al., 2011), self-similar protein dynamics (Glockle and Nonnenmacher, 1995), food science (Rahimy, 2010), food gums (David and Katayama, 2013), fractional dynamics (Tarsov, 2011; Zaslavsky, 2005), quantum dynamics (Iomin, 2009), modeling cardiac tissue electrode interface (Magin, 2008), food engineering and econophysics (David et al., 2011), Hamiltonian chaotic systems (Hilfer, 2000; Zaslavsky, 2005), complex dynamics in biological tissues (Margin, 2010), viscoelasticity (Dalir and Bashour, 2010; Mainardi, 2010; Podlubny, 1999; Rahimy, 2010; Sabatier et al., 2007), control science (Shanantu Das, 2011; Sabatier et al., 2007), quantum mechanics (Herrmann, 2011), modeling oscillation systems (Gomez-Aguilar et al., 2015). Some of these mentioned applications were tried to be touched as follows.

In the area of science and engineering, many applications of fractional calculus have been developed in the last two decades. For instance, fractional calculus was used in image processing, mortgage, biosciences, robotics, motion of fractional oscillator and analytical science (Kumar and Saxena, 2016). It was also used to generalize traditional classical inventory model to fractional inventory model (Das and Roy, 2014).

In the area of electrochemical process, for example half-order derivatives and integrals proved to be more useful for the formulation of certain electrochemical problems than the classical models (Millar and Ross, 1993; Oldham and Spanier, 1974; Podlubny, 1999).

In the area of quantum dynamics, fractional calculus to quantum process was presented. Particularly, the quantum dynamics was considered in the framework of the fractional time Schrodinger equation, which differs from the standard Schrodinger equation by fractional time derivative: $\frac{\partial}{\partial t} \to \frac{\partial^{\alpha}}{\partial t^{\alpha}}$ and it was shown that for $\alpha = \frac{1}{2}$, the fractional Schrodinger equation is isopectral to a comb model (Iomin, 2009). An analytical expression for the Green's functions of the systems was obtained and semi-classical limit was discussed (Iomin, 2009).

In the area of filled polymers, the fractional calculus approach to describe dynamic processes in disordered or complex systems such as relaxation or dielectric behavior in polymers or photo bleaching recovery in biological membranes has proved to be an extraordinarily successful tool (Metzler et al., 1995). Fractional relaxation was applied to filled polymer networks and the dependence of the decisive occurring parameters on the filler content was investigated, and as a result, the dynamics of such complex systems may be well–described by Metzler, Schick, Kilian, and Nonnenmacher fractional model where by the parameters agree with known phenomenological models (Metzler et al., 1995).

In the area of viscoelasticity, the use of fractional calculus for modeling viscoelastic materials is well known. For viscoelastic materials the stress-strain constitutive relation can be more accurately described by introducing the fractional derivative (Carpinteri et al., 2014; Dalir and Bashour, 2010; Duan, 2016; Koeller, 1984; Mainardi, 2010; Podlubny, 1999).

Fractional derivatives, which are the one part of fractional calculus are used to name derivatives of an arbitrary order (Podlubny, 1999). Recently, fractional derivatives have been successfully applied to describe (model) real world problems.

In the area of physics, fractional kinetic equations of the diffusion, diffusion-advection and Focker-Plank type are presented as a useful approach for the description of transport dynamics in complex systems that are governed by anomalous diffusion and non-exponential relaxation patterns (Metzler and Klafter, 2000). Metzler and Klafter (2000) derived these fractional equations asymptotically from basic random walk models, and from a generalized master equation. They presented an integral transformation between the Brownian solution and its fractional counterparts. Moreover, a phase space model was presented to explain the genesis of fractional dynamics in trapping systems. These issues make the fractional equation approach powerful. Their work demonstrates that the fractional equations have come of age as a complementary tool in the description of anomalous transport processes. da Silva et al. (2009) also discussed that solutions for a system governed by a non-Markovian Fokker Planck equation and subjected to a Comb structure were investigated by using the Green function approach. This structure consists of the axis of structure as the backbone and fingers which are attached perpendicular to the axis, and for this system, an arbitrary initial condition in the presence of time dependent diffusion coefficients and spatial fractional derivatives was considered and the connection to the anomalous diffusion was analyzed (da Silva et al., 2009).

In addition to these, the following are also other applications of fractional derivatives. Fractional derivatives in the sense of Caputo fractional derivatives were used in generalizing some theorems of classical power series to fractional power series (El-Ajou et al., 2013). Fractional derivative in the Caputo sense was used to introduce a general form of the generalized Taylor's formula by generalizing some theorems related to the classical power series into fractional power series sense (El-Ajou et al., 2015a). A definition of Caputo fractional derivative proposed on a finite interval in the fractional Sobolev spaces was investigated from the operator theoretic viewpoint (Gorenflo et al., 2015). Particularly, some important equivalence of the norms related to the fractional integration and differentiation operators in the fractional Sobolev spaces are given and then applied for proving the maximal regularity of the solutions to some initial-boundary-value problems for the time-fractional diffusion equation with the Caputo derivative in the fractional Sobolev spaces (Gorenflo et al., 2015). With the help of Caputo time-fractional derivative and Riesz space-fractional derivative, the α-fractional diffusion equation, which is a special model for the two-dimensional anomalous diffusion is deduced from the basic continuous time random walk equations in terms of a time- and space- fractional partial differential equation with the Caputo time-fractional derivative of order $\frac{\alpha}{2}$ and the Riesz space-fractional derivative of order α (Luchko, 2016). Fractional derivatives were also used to describe HIV infection of $CD4^{+}T$ with therapy effect (Zeid et al., 2016).

In the area of modeling oscillating systems, Caputo and Caputo-Fabrizio fractional derivatives were used to present fractional differential equations which are generalization of the classical mass-spring-damper model, and these fractional differential equations are used to describe variety of systems which had not been addressed by the classical mass-spring-damper model due to the limitations of the classical calculus (Gomez-Aguilar et al., 2015).

Podlubny (1999) stated that fractional differential equations are equations which contain fractional derivatives. These equations can be divided into two categories such as fractional ordinary differential equations and fractional partial differential equations. Fractional partial differential equations (FPDEs) are a type of differential equations (DEs) involving multivariable function and their fractional or fractional-integer partial derivatives with respect to those variables (Abu Arqub et al., 2015). There are different examples of fractional partial differential equations. Some of them are: the time-fractional Boussinesq-type equation, the time-fractional $B(2,1,1)$ -type equation and the time-fractional Klein-Gordon-type equation stated in Abu Arqub et al. (2015), and time fractional diffusion equation stated in Kumar et al. (2017), Cetinkaya and Kiymaz (2013), Kumar et al. (2012) and so on.

Recently, fractional differential equations have been successfully applied to describe (model) real world problems. For instance, the generalized wave equation, which contains fractional derivatives with respect to time in addition to the second-order temporal and spatial derivatives, was used to model the viscoelastic case and the pure elastic case in a single equation (Wang, 2016). The time fractional Boussinesq-type equations can be used to describe small oscillations of nonlinear beams, long waves over an even slope, shallow-water waves, shallow fluid layers, and nonlinear atomic chains; the time-fractional $B(2,1,1)$ -type equations can be used to study optical solitons in the two cycle regime, density waves in traffic flow of two kinds of vehicles, and surface acoustic soliton in a system supporting love waves; the time fractional Klein-Gordon-type equations can be applied to study complex group velocity and energy transport in absorbing media, short waves in nonlinear dispersive models, propagation of dislocations within crystals (Abu Arqub et al., 2015). According to Abu Arqub (2017), the time-fractional heat equation, which is derived from Fourier's law and conservation of energy, is used in describing the distribution of heat or variation in temperature in a given region over time; the time-fractional cable equation, which is derived from the cable equation for electro diffusion in smooth homogeneous cylinders and occurred due to anomalous diffusion, is used in modeling the ion electro diffusion at the neurons; the time-fractional modified anomalous sub diffusion equation, which is derived from the neural cell adhesion molecules is used for describing processes that become less anomalous as time progresses by the inclusion of a second fractional time derivative acting on the diffusion term; the time fractional reaction sub diffusion equation is used to describe many different areas of chemical reactions, such as exciton quenching, recombination of charge carriers or radiation defects in solids, and predator pray relationships in ecology. The time-fractional Fokker–Planck equation is used to describe many phenomena in plasma and polymer physics, population dynamics, neurosciences, nonlinear hydrodynamics, pattern formation, and psychology. The time-fractional Fisher's equation is used to describe the population growth models, whilst, the time fractional Newell–Whitehead equation is used to describe fluid dynamics model and capillary–gravity waves. The fractional differential equations, generalization of the classical mass-spring-damper models are useful in understanding the behavior of dynamical complex systems, mechanical vibrations, control theory, relaxation phenomena, viscoelasticity, viscoelastic damping and oscillatory processes (Gomez-Aguilar et al., 2015). The space-time fractional diffusion equations on two time intervals was used in finance to model option pricing and the model was shown to be useful for option pricing during some temporally abnormal periods (Korbel and Luchko, 2016). The α -fractional diffusion equation for $0<\alpha<2$ describes the so called Levy flights that correspond to the continuous time random walk model, where both the mean waiting time and the jump length variance of the diffusing Particles are divergent (Luchko, 2016). Time fractional diffusion equations in the Caputo sense with initial conditions are used to model cancer tumor (Iyiola and Zaman, 2014).

Nonlinear diffusion equations play a great role to describe the density dynamics in a material undergoing diffusion in a dynamic system which includes different branches of science and technology. The classical and simplest diffusion equation which is used to model the free motion of the particle is:

$$\frac{\partial u(x,t)}{\partial t}=A\frac{\partial^2}{\partial x^2}u(x,t)-\frac{\partial}{\partial x}\big(F(x)u(x,t)\big),\ \mathrm{A}>0 \qquad (1),$$

where $u(x,t)$ is the probability density function of finding a particle at the point x in time instant t , $F(x)$ is the external force, and A is a positive constant which depends on the temperature, the friction coefficient, the universal gas constant and the Avogadro number (Kumar et al., 2017).

Recently, the fractional differential equations have gained much attention of researchers due to the fact that they generate fractional Brownian motion which is generalization of Brownian motion (Podlubny, 1999). Das et al. (2011) stated that time fractional diffusion equation, which is one of the fractional differential equations, is obtained from the classical diffusion equation in mathematical physics by replacing the first order time derivative by a fractional derivative of order α where $0<\alpha<1$. Time fractional diffusion equation is an evolution equation that generates the fractional Brownian motion (FBM) which is a generalization of Brownian motion (Das et al., 2011; Podlubny, 1999). Due to the fact that fractional derivative provides an excellent tool for describing memory and hereditary properties for various materials and processes (Caputo and Mainardi, 1971), the time fractional diffusion equation (Kumar et al., 2017; Cetinkaya and Kiymaz, 2013; Das, 2009; Kumar et al., 2012) of the form

$$\begin{cases}\dfrac{\partial^\beta u(x,t)}{\partial t^\beta}=\lambda\dfrac{\partial^2}{\partial x^2}u(x,t)-\dfrac{\partial}{\partial x}\big(F(x)u(x,t)\big),\ 0<\beta\le 1,\ x>0,\ t>0\ ,\ \lambda>0 & \text{(2a)}\\ \text{subject to initial condition (IC)}:\ u(x,0)=f(x) & \text{(2b)}\end{cases}$$

which is generalization of Equation 1, was considered in this study. Here $D_t^\beta u(x,t)=J_t^{1-\beta}\left[\dfrac{\partial}{\partial t}u(x,t)\right]$.

The fractional derivative D_t^β is considered in the Caputo sense which has the main advantage that the initial conditions for fractional differential equations with Caputo derivative take on the same form as for integer order differential equations (Caputo, 1967). Due to this, considerable works on fractional diffusion equations have already been done by different authors to obtain exact, approximate analytic and pure numerical solutions by using various developed methods.

Recently, Adomian Decomposition Method by Saha Ray and Bera in 2006 (as cited in Cetinkaya and Kiymaz, 2013; Kumar et al., 2012; Das, 2009), variational iteration method (Das, 2009), Homotopy Analysis Method (Das et al., 2011), Laplace Transform Method (Kumar et al., 2012), Generalized Differential Transform Method (Cetinkaya and Kiymaz, 2013) and Residue fractional power series method (Kumar et al., 2017), which have their own inbuilt deficiencies: the complexity and difficulty of solution procedure for calculation of Adomain polynomials, the restrictions on the order of the nonlinearity term or even the form of the boundary conditions and uncontrollability of non-zero end conditions, unrestricted freedom to choose base function to approximate the linear and nonlinear problems, and complex computations respectively, were used to obtain solutions of time fractional diffusion equations with initial conditions.

To overcome these so called deficiencies, the reduced differential transform method (Keskin and Outranc, 2009, 2010) in its fractional form was preferably taken in this paper to solve time fractional diffusion equations with initial conditions of the form (2a) given that (2b) analytically with the help of the general recurrence relation (that is, Equation (24a) with (24b) which was developed in this paper.

The fractional reduced differential transform technique is an iterative procedure for obtaining series solution of differential equations and it reduces the size of computational work and easily applicable to many physical problems (Mohyud-Din and Sohail, 2012a, b). Recently, fractional reduced differential transform method was used to solve different time fractional partial differential equations. For example, fractional reduced differential transform method was used by Jafari et al. (2016) to solve partial differential equations within local fractional derivative operators. Fractional reduced differential transform method was used by Singh et al. (2016) to solve time-Fractional order Black-Scholes option pricing equation. Fractional reduced differential transform method was also used in solving two and three dimensional time fractional telegraphic equations (Srivastava et al., 2014), time fractional nonlinear evolution equations (Abdou and Yildirim, 2012) and Caputo time fractional order hyperbolic telegraph equation (Sirvastava et al., 2013).

In this paper, the author has examined a general recurrence relation by the use of fractional reduced differential transform and then a scheme to find closed solutions of one dimensional time fractional diffusion equations with initial conditions in the form of infinite fractional power series and in terms of Mittag-Leffler function in one parameter as well as their exact solutions by the use of fractional reduced differential transform method. The results of the listed examples showed that the general recurrence relation works successfully in solving time-fractional diffusion equations with initial conditions in a direct way without using linearization, transformation, perturbation, discretization or restrictive assumptions by using Fractional Reduced Differential Transform method.

This paper is organized as follows: First is the methodology, followed by results and discussion involving: some definitions, theorems and notations of fractional calculus theory, the results which are the new recurrence relation and methodology on how to apply fractional reduced differential transform method, application models and discussion of application of the results obtained were presented. Finally, conclusions are presented.

METHODOLOGY

In this paper, it was designed to set and discuss the theoretical background (foundation) of the study step by step to come to the objective of the study. Next, it was designed to consider time fractional differential equations under initial conditions, specifically, time fractional diffusion equations with initial conditions of the form (2a) given that (2b), and then use analytical design to solve them analytically by using fractional reduced differential transform method with the help of a new general recurrence relation which can be obtained from fractional reduced transform by following the next five procedures sequentially. First, it was designed to discuss some definitions (specially, the definition of Caputo fractional derivative), properties, lemma and theorems of fractional calculus, and definitions and some theorems of fractional reduced differential transform, which were used in the study. Secondly, it was designed to develop a new recurrence relation and then methodology of fractional reduced differential transform method for (2a) given that (2b). Thirdly, it was designed to obtain closed solutions of (2a) given that (2b) in the form of infinite fractional power series by fractional reduced differential transform method with the help of the recurrence relation of (2a) given that (2b). Fourthly, it was designed to determine closed solutions of (2a) given that (2b) in terms of Mittag-Leffler functions in one parameter from these infinite fractional power series closed form solutions of (2a) given that (2b). Lastly, it was designed to obtain exact solutions from closed solutions in terms of Mittag-Leffler functions in one parameter of (2a) given that (2b) for the special case $\alpha = 1$.

RESULTS AND DISCUSSION

Preliminaries

Fractional calculus

There are several definitions of both the fractional integration of order $\beta \geq 0$ and the fractional derivative of order $\beta \geq 0$, not automatically equivalent to each other (Millar and Ross, 1993). The two most used definitions, Riemann-Liouville and Caputo definitions and some properties of fractional calculus are revisited as follow to use them in this paper (Kilbas et al., 2006; Mainardi, 2010; Podlubny, 1999; Millar and Ross, 1993).

Definition 1: A real valued function $u(x,t), x \in IR, t > 0$, is said to be in the space $C_\mu, \mu \in IR$, if there exists a real number $q > \mu$ such that $u(x) = t^q u_1(x,t)$, where $u_1(x,t) \in C(IR \times [0,+\infty))$ and it is said to be in the space C_μ^m if $u^{(m)}(x,t) \in C_\mu, n \in IN$.

Definition 2: The Riemann-Liouville fractional integral operator of order $\beta \geq 0$ of a function $u(x,t) \in C_\mu, \mu > -1$ is defined as

$$J_t^\beta u(x,t) = \begin{cases} \frac{1}{\Gamma(\beta)} \int_0^t (x-\xi)^{\beta-1} u(x,\xi) d\xi, \ 0 < \xi < \mathrm{t}, \beta > 0 \\ u(x,t), \beta = 0 \end{cases} \quad (3)$$

Consequently, for

$$\alpha, \beta \geq 0, C \in IR, u(x,t) \in C_\mu^m, u(x,t) \in C_\mu, \mu > -1,$$

the operator J_t^β has the following properties:

1. $J_t^\alpha J_t^\beta u(x,t) = J_t^{\alpha+\beta} u(x,t) = J_t^\beta J_t^\alpha u(x,t)$

2. $J_t^{\alpha} c = \left(\frac{c}{\Gamma(\alpha+1)} \right) t^{\alpha}$

3. $J_t^{\alpha} t^{\gamma} = \left(\frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1+\alpha)} \right) t^{\gamma+\alpha}$

The Riemann Liouville derivative has the disadvantage that it does not allow utilization of initial and boundary conditions involving integer order derivatives when trying to model real world problems with fractional differential equations. To beat this disadvantage of Riemann Liouville derivative (Millar and Ross, 1993; Podlubny, 1999), Caputo proposed a modified fractional differentiation operator D_a^{β} (Caputo and Mainardi, 1971) to illustrate the theory of viscoelasticity as follows:

$$D_a^{\beta} f(x) = J_a^{m-\beta} D^m f(x) = \frac{1}{\Gamma(m-\beta)} \int_a^x (x-\xi)^{m-\beta-1} f^{(m)}(\xi) d\xi, \beta \geq 0$$

for $m-1 < \beta < m, x > a$ and $f \in C_{-1}^m$.

This Caputo fractional derivative allows the utilization of initial and boundary conditions involving integer order derivatives, which have clear physical interpretations of the real situations.

Definition 3: For the smallest integer that exceeds β, the Caputo time fractional derivative order $\beta > 0$ of a function $u(x,t)$ is defined as:

$$D_t^{\beta} u(x,t) = \begin{cases} \frac{1}{\Gamma(m-\beta)} \int_0^t (t-\xi)^{m-\beta-1} \frac{\partial^m u(x,\xi)}{\partial \xi^m} d\xi \\ \frac{\partial^m u(x,t)}{\partial t^m}, \beta = m \end{cases} = J^{m-\beta} \frac{d^m}{dt^m} u(x,t),\ 0 \leq m-1 < \beta < m \quad (4)$$

Theorem 1: If $m-1 < \beta \leq m$, $\forall \mathrm{m} \in \mathrm{IN}, u(x,t) \in C_{\gamma}^m, \gamma \geq -1$ then

$$D_t^{\beta} J_t^{\beta} u(x,t) = u(x,t). \quad \text{(i)}$$

$$J^{\beta} D^{\beta} u(x,t) = u(x,t) - \sum_{k=0}^{m-1} \frac{\partial^k}{\partial t^k} u(x,0^+) \frac{t^k}{k!},\ t > 0. \quad \text{(ii)}$$

The reader is kindly requested to go through (Kilbas et al., 2006; Mainardi, 2010) in order to know more details about the mathematical properties of fractional derivatives and fractional integrals, including their types and history, their motivation for use, their characteristics, and their applications.

According to Omez-Aguilar et al. (2014) and the reference therein, the Caputo fractional derivative operator D^{β} satisfies the linearity property:

$${}_0^C D_x^{\beta} [f(x) + g(x)] = {}_0^C D_x^{\beta} [f(x)] + {}_0^C D_x^{\beta} [g(x)] \quad (5)$$

Definition 4: According to Millar and Ross (1993), Podlubny (1999) and Sontakke and Shaikh (2015), the Mittag-Leffler function, which is a one parameter generalization of exponential function, is defined as

$$E_{\alpha}(z) = \sum_{q=0}^{\infty} \frac{z^q}{\Gamma(q\alpha+1)}, \alpha \in C, \mathrm{Re}(\alpha) > 0 \quad (6)$$

Definition 5: According to Neog (2015), a Taylor series of a polynomial of degree n is defined as follows:

$$f_n(x) = \sum_{n=0}^{n} \frac{f^{(n)}(c)}{n!} (x-c)^n \quad (7)$$

Theorem 2: According to Neog (2015), if the function has $(n+1)$ derivatives on an interval $(c-r, c+r)$ for some $r > 0$, and $f(x)$ $\lim_{\substack{n\to\infty \\ n \in IN}} R_n(x) = 0$, $\forall x (c-r, c+r)$ is the error between and the polynomial function then the Taylor series expanded about c converges to $f(x)$. Thus,

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!} (x-c)^n, \forall x (c-r, c+r) \quad (8)$$

Definition 6: According to El-Ajou et al. (2013), a power series expansion of the form:

$$\sum_{n=0}^{\infty} c_n (t-t_0)^{n\beta} = c_0 + c_1 (t-t_0)^{\beta} + c_2 (t-t_0)^{2\beta} + c_3 (t-t_0)^{3\beta} + \cdots, 0 \leq m-1 < \beta \leq m, m \in IN \text{ and } t \geq t_0 \quad (9)$$

is called fractional power series about t_0, where t is a variable and c_m is called coefficients of the series.

Theorem 3: According to El-Ajou et al. (2013), suppose that $f(t)$ has fractional power series representation at $t = t_0$, then

$$f(t) = \sum_{n=0}^{\infty} c_n (t-t_0)^{n\beta}, 0 \leq m-1 < \beta \leq m, m \in IN, t_0 \leq \mathrm{t} < \mathrm{t}_0 + \mathrm{R} \quad (10)$$

If $D^{n\beta} f(t)$ are continuous on $(t_0, t_0 + R), n = 0, 1, 2, \cdots,$

then the coefficients c_n of Equation (10) are given by the formula

$$c_n = \frac{D^{n\beta} f(t_0)}{\Gamma(n\beta+1)}, n = 0, 1, 2, \cdots \tag{11}$$

where $D^{n\beta} = D^{\beta}.D^{\beta}.D^{\beta}.\cdots.D^{\beta}$ (m-times) and R is the radius of convergence.

Remark 1: By substituting the form of c_n of Equation (11), we can notice that the fractional power series expansion of $f(t)$ about $t = t_0$ must be of the form

$$f(t) = \sum_{n=0}^{\infty} \frac{D^{n\beta} f(t_0)}{\Gamma(n\beta+1)} (t-t_0)^{n\beta}, 0 \le m-1 < \beta \le m, m \in IN, x \in I \subseteq \mathrm{IR}, t_0 \le \mathrm{t} < \mathrm{t}_0 + \mathrm{R} \tag{12}$$

which is generalized Taylor's series formula. To be specific, if one set $\alpha = 1$, then the classical Taylor's series formula

$$f(t) = \sum_{n=0}^{\infty} \frac{D^n f(t_0)}{n!} (t-t_0)^n, x \in I \subseteq \mathrm{IR}, t_0 \le \mathrm{t} < \mathrm{t}_0 + \mathrm{R} \tag{13}$$

The following definition and the theorem with its proof were given in El-Ajou et al. (2015b).

Definition 7: If $0 \le m-1 < \beta \le m, m \in IN \text{ and } t \ge t_0$ a fractional power series of the form

$$\sum_{n=0}^{\infty} f_n(x)(t-t_0)^{n\beta} = f_0(x) + f_1(x)(t-t_0)^{\beta} + f_2(x)(t-t_0)^{2\beta} + \cdots, x \in I \subseteq IR, \mathrm{t} \ge t_0 \tag{14}$$

is called a multiple fractional power series about $t = t_0$ where t is a variable and $f_m(x)$ are function of x called the coefficients of the series.

Theorem 4: Suppose that $u(x,t)$ has a multiple fractional power series representation at $t = t_0$ of the form

$$u(x,t) = \sum_{n=0}^{\infty} f_n(x)(t-t_0)^{n\beta}, 0 \le m-1 < \beta \le m, m \in IN, t_0 \le \mathrm{t} < \mathrm{t}_0 + \mathrm{R} \tag{15}$$

If $D_t^{n\beta} u(x,t)$ are continuous on $I \times (t_0, t_0 + R), n = 0, 1, 2, \cdots$, then the coefficients $f_n(x)$ of Equation (15) are given by the formula

$$f_n(x) = \frac{D^{n\beta} u(x, t_0)}{\Gamma(n\beta+1)}, n = 0, 1, 2, \cdots \tag{16}$$

where $D_t^{n\beta} = \frac{\partial^{\beta}}{\partial t^{\beta}} . \frac{\partial^{\beta}}{\partial t^{\beta}} \cdots \frac{\partial^{\beta}}{\partial t^{\beta}}$ (m-times) and $R = \min_{c \in I} R_C$ is the radius of convergence in which R_C is the radius of convergence of the fractional power series $\sum_{n=0}^{\infty} f_n(c)(t-t_0)^{n\beta}$.

Remark 2: By substituting the form of $f_n(x)$ of Equation (16), we can notice that the multiple fractional power series expansion of $u(x,t)$ about $t = t_0$ must be of the form

$$u(x,t) = \sum_{n=0}^{\infty} \frac{D^{n\beta} u(x,t_0)}{\Gamma(n\beta+1)} (t-t_0)^{n\beta}, 0 \le m-1 < \beta \le m, m \in IN, x \in I \subseteq \mathrm{IR}, t_0 \le \mathrm{t} < \mathrm{t}_0 + \mathrm{R} \tag{17}$$

which is generalized Taylor's series formula. To be specific, if one set $\alpha = 1$, then the classical Taylor's series formula

$$u(x,t) = \sum_{n=0}^{\infty} \frac{D^n u(x,t_0)}{n!} (t-t_0)^n, x \in I \subseteq \mathrm{IR}, t_0 \le \mathrm{t} < \mathrm{t}_0 + \mathrm{R} \tag{18}$$

Fractional reduced differential transform

Here, the basic definitions of fractional reduced differential transformations were introduced.

Consider a function of two variables, $u(x,t)$ such that $u(x,t) = q(x)g(t)$, where $q(x)$ and $g(t)$ are analytic and k times continuously differentiable with respect to variable x and $g(t)$ is analytic and k times continuously differentiable with β^{th} derivatives with respect to time t where $\beta > 0$. Then based on Equations (8 and 12), and the properties of one dimensional differential transformation (Hilfer, 2000; Keskin and Outranc, 2009), the function $u(x,t)$ can be represented as

$$\begin{aligned} u(x,t) &= \sum_{i=0}^{\infty} Q(i)(x-x_0)^i \sum_{j=0}^{\infty} G_{\beta}(j)(t-t_0)^{j\beta} = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} Q(i)G_{\beta}(j)x^i t^{j\beta} = \sum_{k=0}^{\infty} \frac{q^{(k)}(x_0)}{k!}(x-x_0)^k \frac{\left(D_{t_0}^{\beta}\right)^k g(t_0)}{\Gamma(k\alpha+1)}(t-t_0)^{k\beta} \\ &= \sum_{k=0}^{\infty} Q(k)(x-x_0)^k \frac{\left(D_{t_0}^{\beta}\right)^k g(t_0)}{\Gamma(k\alpha+1)}(t-t_0)^{j\beta} = \sum_{k=0}^{\infty} q(x) \frac{\left(D_{t_0}^{\beta}\right)^k g(t_0)}{\Gamma(k\alpha+1)}(t-t_0)^{j\beta} = \sum_{k=0}^{\infty} \frac{\left(D_{t_0}^{\beta}\right)^k q(x)g(t_0)}{\Gamma(k\alpha+1)}(t-t_0)^{j\beta} \\ &= \sum_{k=0}^{\infty} \left(\underbrace{\left[\frac{\left(D_{t_0}^{\beta}\right)^k u(x,t)}{\Gamma(k\alpha+1)} \right]_{t=t_0}}_{U_k(x)} \right)(t-t_0)^{j\beta} = \sum_{k=0}^{\infty} U_k(x)(t-t_0)^{k\beta}, \end{aligned} \tag{19}$$

where $U_k(x)$ is the t-dimensional spectrum function of $u(x,t)$.

For example, consider the function $u(x,t) = e^{x-x_0} E_{\beta}\left((t-t_0)^{\beta}\right)$. This function can be written as:

$$u(x,t)=e^{x-x_0}\left(\underbrace{1+\frac{(t-t_0)^{\beta}}{\Gamma(\beta+1)}+\frac{(t-t_0)^{2\beta}}{\Gamma(2\beta+1)}+\cdots}_{E_\beta\left((t-t_0)^\beta\right)}\right)=e^{x-x_0}+e^{x-x_0}\,\frac{(t-t_0)^{\beta}}{\Gamma(\beta+1)}+e^{x-x_0}\,\frac{(t-t_0)^{2\beta}}{\Gamma(2\beta+1)}+\cdots=\sum_{k=0}^{\infty}\left(\underbrace{\frac{e^{x-x_0}}{\Gamma(2\beta+1)}}_{U_k(x)}\right)(t-t_0)^{k\beta}=\sum_{k=0}^{\infty}U_k(x)(t-t_0)^{k\beta}$$

In this paper, $FRDT$ and $FRDT^{-1}$ denote the fractional reduced differential transform operator and inverse fractional reduced differential transform operator respectively.

So, based on Equations (16 and 19) and (17 and 19), the definition of fractional reduced differential transform (FRDT) of a function $u(x,t)$ denoted by $U_k(x)$ and the definition of the differential inverse fractional reduced transform of $U_k(x)$ denoted by $u(x,t)$ were given respectively as follows.

Definition 8: If the function $u(x,t)$ such that $u(x,t)=q(x)g(t)$ is analytic and k-times continuously differentiable with respect to space variable x and also k times continuously differentiable with β^{th} derivative with respect to time t in the domain of interest, then the fractional reduced differential transform (FRDT) of a function $u(x,t)$ (the $t-$ dimensional spectrum function of $u(x,t)$) denoted by $U_K(x)$ is given by:

$$FRDT[u(x,t)]=U_K(x)=\frac{1}{\Gamma(k\beta+1)}\left[\left(D_{t_0}^{\beta}\right)^k u(x,t)\right]_{t=t_0}=\frac{1}{\Gamma(k\beta+1)}\left[\frac{\partial^{k\beta}}{\partial t^{k\beta}}u(x,t)\right]_{t=t_0} \qquad (20)$$

where β such that $0<\beta\le 1$ is a parameter describing the order of the time fractional derivative in Caputo sense.

Definition 9: The differential inverse fractional reduced transform of $U_k(x)$ denoted by $u(x,t)$ is given by:

$$FRDT^{-1}(U_k(x))=u(x,t)=\sum_{k=0}^{\infty}U_K(x)(t-t_0)^{k\beta} \qquad (21)$$

where β such that $0<\beta\le 1$ is a parameter describing the order of the time fractional derivative in Caputo sense.

Combining Equations (20 and 21) one can obtain:

$$u(x,t)=\sum_{k=0}^{\infty}\frac{1}{\Gamma(k\beta+1)}\left[\frac{\partial^{k\beta}}{\partial t^{k\beta}}u(x,t)\right]_{t=t_0}(t-t_0)^{k\beta} \qquad (22)$$

which is the differential inverse fractional reduced transform (Equation (20)).

Equation (22) confirms that the concept of the fractional reduced differential transform is derived from the fractional power series expansion of the function $u(x,t)$.

Remark 3: In real application, the function $u(x,t)$ is represented by a finite series of Equation (21), which can be expressed as:

$$u(x,t)=\sum_{k=0}^{n}U_K(x)(t-t_0)^{k\beta}+R_n(x,t) \qquad (23)$$

and Equation (21) implies that $R_n(x,t)=\sum_{k=n+1}^{\infty}U_K(x)(t-t_0)^{k\beta}$ negligibly small. Usually, the values of n is decided by convergence of the series coefficients.

Even though there are different theorems which can be deduced from Equations (20) and (21), Theorems 5 to 8 (Keskin and Outranc, 2009, 2010) and Theorem 9 (Mohyud-Din and Sohail, 2012a, b) are the ones which were revisited to use in this study.

Theorem 5: If $w(x,t)=v(x,t)\pm u(x,t)$, then $W_k(x)=V_k(x,t)\pm U_k(x)$.

Theorem 6: If $w(x,t)=\alpha u(x,t)$, then $W_k(x)=\alpha U_k(x)$, α is a constant and $k=0,1,\ 2,\cdots$

Theorem 7: If $w(x,t)=\frac{\partial}{\partial x}u(x,t)$, then $W_k(x)=\frac{\partial}{\partial x}U_k(x)$, $k=0,1,\ 2,\cdots$

Theorem 8: If $w(x,t)=\frac{\partial^2}{\partial x^2}u(x,t)$, then

$$W_k(x) = \frac{\partial^2}{\partial x^2} U_k(x), \; k = 0, 1, \; 2, \cdots$$

Theorem 9: If $w(x,t) = \dfrac{\partial^{N\beta}}{\partial t^{N\beta}} u(x,t)$, then

$$W_k(x) = \frac{\Gamma((\beta k + N\beta + 1)}{\Gamma((\beta k + 1)} U_{k+N}(x),$$
$$0 < \beta \le 1, k = 1, \; 2, \; \cdots.$$

Main results

Here, a new general recurrence relation, and methodology of fractional reduced differential transform method were developed and introduced for determining closed solutions of time fractional diffusion equations of the form (2a) given that (2b) in infinite fractional power series form, in terms of Mittag-Leffler functions in one parameter and exact form.

Solution of the general problem

Here, as a result of using Equations (20) and (21) and Theorems 5 to 9, the new general recurrence relation for fractional reduced differential transform of (2a) given that (2b) was developed. With the help of it, a methodology of fractional reduced differential transforms method, which is used to obtain closed solutions of (2a) given that (2b) in infinite fractional power series and in terms of Mittag-Leffler function in one parameter as well as their exact solutions, was developed in analytic form.

The general recurrence relation:

Theorem 10: If the function $F(x)$ has the Taylor series at a, which is $F(x) = \sum_{n=0}^{\infty} \frac{F^{(n)}(a)}{n!}(x-a)^n$ with a radius of convergence $R > |x-a| > 0$, then the general recurrence relation which can be obtained from the fractional reduced differential transform (FRDT) of Equation (2a) given that (2b) is given by

$$\begin{cases} U_{k+1}(x) = \dfrac{\Gamma(k\alpha+1)}{\Gamma(k\alpha+\alpha+1)}\left(\lambda \dfrac{\partial^2}{\partial x^2} U_k(x) + \left(\dfrac{\partial}{\partial x} F(x)\right) U_k(x) + F(x)\left(\dfrac{\partial}{\partial x} U_k(x)\right)\right), 0 < \beta \le 1, \mathrm{x} > 0, t > 0, k = 0, 1, 2, \cdots & (24a) \\ U_0(x) = f(x), \; k = 0 & (24b) \end{cases}$$

Proof:

By taking $FRDT$ of both sides of (2a) where $F(x)$ has the Taylor series at a, which is $F(x) = \sum_{n=0}^{\infty} \frac{F^{(n)}(a)}{n!}(x-a)^n$ with a radius of convergence $R > |x-a| > 0$ and then using Theorem 9 and Equation (20), we have:

$$\frac{\Gamma(k\beta+\beta+1)}{\Gamma(k\beta+1)} U_{k+1}(x) = \frac{1}{\Gamma(k\alpha+1)}\left[\frac{\partial^{\beta k}}{\partial t^{\beta k}}\left(\lambda \frac{\partial^2}{\partial x^2} u - \frac{\partial}{\partial x}\left(F(x)u(x,t)\right)\right)\right]_{t_0=0}, \; 0 < \beta \le 1, \mathrm{x} > 0, t > 0, k = 0, 1, 2, \cdots$$

By Equation (5),

$$\frac{\Gamma(k\beta+\beta+1)}{\Gamma(k\beta+1)} U_{k+1}(x) = \frac{1}{\Gamma(k\alpha+1)}\left[\frac{\partial^{\beta k}}{\partial t^{\beta k}}\left(\lambda \frac{\partial^2}{\partial x^2} u\right) - \frac{\partial^{\beta k}}{\partial t^{\beta k}}\left(\frac{\partial}{\partial x}\left(F(x)u(x,t)\right)\right)\right]_{t_0=0}, \; 0 < \beta \le 1, \mathrm{x} > 0, t > 0, k = 0, 1, 2, \cdots$$

Since multiplication is distributive over subtraction,

$$\frac{\Gamma(k\beta+\beta+1)}{\Gamma(k\beta+1)} U_{k+1}(x) = \frac{1}{\Gamma(k\alpha+1)}\left[\frac{\partial^{\beta k}}{\partial t^{\beta k}}\left(\lambda \frac{\partial^2}{\partial x^2} u\right)\right]_{t_0=0} - \frac{1}{\Gamma(k\alpha+1)}\left[\frac{\partial^{\beta k}}{\partial t^{\beta k}}\left(\frac{\partial}{\partial x}\left(F(x)u(x,t)\right)\right)\right]_{t_0=0}, 0 < \beta \le 1, \mathrm{x} > 0, t > 0, k = 0, 1, 2, \cdots$$

By Theorems 5 to 9,

$$\frac{\Gamma(k\beta+\beta+1)}{\Gamma(k\beta+1)} U_{k+1}(x) = \lambda \frac{\partial^2}{\partial x^2} U_k(x) - \left(\left(\frac{\partial}{\partial x} F(x)\right) U_k(x) + F(x)\left(\frac{\partial}{\partial x} U_k(x)\right)\right), \; 0 < \beta \le 1, \mathrm{x} > 0, t > 0, k = 0, 1, 2, \cdots$$

Thus, the FRDT of $\dfrac{\partial^\beta u}{\partial t^\beta} = \lambda \dfrac{\partial^2}{\partial x^2} u - \dfrac{\partial}{\partial x}\left(F(x)u(x,t)\right), 0 < \beta \le 1, t > 0$ is given by:

$$U_{k+1}(x) = \frac{\Gamma(k\alpha+1)}{\Gamma(k\alpha+\alpha+1)}\left(\lambda \frac{\partial^2}{\partial x^2} U_k(x) - \left(\left(\frac{\partial}{\partial x} F(x)\right) U_k(x) + F(x)\left(\frac{\partial}{\partial x} U_k(x)\right)\right)\right), \; 0 < \beta \le 1, t > 0, k = 0, 1, 2, \cdots \quad (24a)$$

Again, by taking $FRDT$ of both sides of (2b) and then using Equation (20).

$$U_K(x) = \frac{1}{\Gamma(k\beta+1)}\left[\frac{\partial^{k\beta}}{\partial t^{k\beta}} u(x,0)\right]_{t_0=0} = \frac{1}{\Gamma(k\beta+1)}\left[\frac{\partial^{k\beta}}{\partial t^{k\beta}} f(x)\right]_{t_0=0}, 0 < \beta \le 1.$$

Since initial condition (at $t_0 = 0$) is where the reduced differential transform starts, that is, $k = 0$, $FRDT[u(x,0)] = U_0(x) = u(x,0) = f(x)$.

Therefore, the FRDT of (2b) is given by

$$U_0(x) = u(x,0) = f(x) \quad (24b)$$

Therefore, the FRDT of Equation (2a) given that (2b) are (24a) and (24b) respectively.

Hence, this completes the proof of the theorem.

Methodology of fractional reduced differential transform method: The overall steps, which were introduced to be followed by anyone accordingly to obtain closed form solutions of (2a) given that (2b) in the form of fractional power series, in terms of Mittag-Leffler function of one parameter and exact form by fractional reduced differential transform method with the help of Theorem 5 are steps 1 to 3, 1 to 4 and 1 to 5, respectively.

Step 1: Using Theorem 5, find the general recurrence relation of problems of the form (2a) given that (2b) and then substitute $U_0(x)$ value in the obtained general recurrence relation successively, to find the other $U_k(x)$ values:

$$U_1(x), U_2(x), U_3(x), \cdots, U_n(x), \forall k = 1, 2, 3, \cdots, n.$$

Step 2: Find the n^{th} order approximate solution of (2a) given that (2b), that is, $\tilde{u}_n(x,t)$ by taking the differential inverse fractional reduced transform of $\{U_k(x)\}_0^n$, which is given by

$$\tilde{u}_n(x,t) = \sum_{k=0}^{n} U_k(x) t^{k\beta}, 0 < \beta \le 1, x > 0, \text{ t} > 0 .$$

Step 3: Letting n to ∞, $\tilde{u}_n(x,t) = \sum_{k=0}^{n} U_k(x) t^{k\beta}, 0 < \beta \le 1, x > 0$ becomes

$$u(x,t) = \lim_{\substack{n\to\infty \\ n\in W}} \tilde{u}_n(x,t) = \sum_{k=0}^{\infty} U_k(x) t^{k\beta}, 0 < \beta \le 1, x > 0 ,$$

which is closed solutions of (2a) given that (2b) in the form of infinite fractional power series.

Step 4: By using Equation (6) in $u(x,t) = \sum_{k=0}^{\infty} U_k(x) t^{k\beta}, 0 < \beta \le 1, x > 0$ of Step 3, we can obtain closed solutions of (2a) given that (2b) in terms of Mittag-Leffler function in one parameter.

Step 5: By taking left hand side limit of result of Step 4 as β approaches 1 from left, that is,

$$\lim_{\beta\to 1^-} \left(u(x,t)\right) = \lim_{\beta\to 1^-} \sum_{k=0}^{\infty} U_k(x) t^{k\beta}, 0 < \beta \le 1, x > 0 ,$$ the exact solutions of (2a) given that (2b), $u_{exact}(x,t)$ can be obtained.

Applications

Here, to validate (show) the simplicity, effectiveness and applicability of the newly proposed recurrence relation (Theorem 5) in the methodology of the fractional reduced differential transform method (FRDTM), five application examples were considered and solved as follow.

Example 1: Taking $F(x) = -x, \lambda = 1$ in (2a) and choosing $f(x) = 1$ in (2b) (Kumar et al., 2012, 2017; Cetinkaya and Kiymaz, 2013), consider the initial value problem:

$$\begin{cases} \dfrac{\partial^\beta u}{\partial t^\beta} = \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial}{\partial x}(xu), x > 0, \text{ t} > 0, 0 < \beta \le 1 & (25a) \\ \text{Subject to initial condition } : u(x,0) = 1 & (25b) \end{cases}$$

Since $F(x) = -x, \lambda = 1$ and $f(x) = 1$, by Theorem 5, the FRDT of Equation (25a) and (25b) are:

$$\begin{cases} U_{k+1}(x) = \dfrac{\Gamma(k\beta+1)}{\Gamma(k\beta+\alpha+1)}\left(\dfrac{\partial^2}{\partial x^2}U_k(x) - (-U_k(x)) - \left(-x\left(\dfrac{\partial}{\partial x}U_k(x)\right)\right)\right), \quad 0 < \beta \le 1, x > 0, t > 0, k = 0, 1, 2, \cdots & (26a) \\ U_0(x) = 1, k = 0 & (26b) \end{cases}$$

By substituting 1 for $U_0(x)$ from Equation (26b) in (26a) recursively, we find the following $U_k(x)$ values:

$$U_1(x), U_2(x), U_3(x), \cdots, U_n(x), \forall k = 1, 2, 3, \cdots, n$$

by straight forward iteration calculation.

For $k = 0; U_1(x) = \dfrac{1}{\Gamma(\beta+1)}, \quad 0 < \beta \le 1, x > 0, t > 0$

For $k = 1; U_2(x) = \dfrac{1}{\Gamma(2\beta+1)}, \quad 0 < \beta \le 1, x > 0, t > 0$

For $k = 2; U_3(x) = \dfrac{1}{\Gamma(3\beta+1)}, \quad 0 < \beta \le 1, x > 0, t > 0$

Continuing with this process, we obtain that:

$$U_k(x) = \frac{1}{\Gamma(k\beta+1)}, \quad 0 < \beta \le 1, x > 0, t > 0, \forall k = 0,1,2,3,\cdots,n \,\&\, \text{IN} \cdot$$

Then the inverse FRDT of $\{U_k(x)\}_{k=0}^n$ is the n^{th} order approximate solution of Equation (25a), $\tilde{u}_n(x,t)$ which is given by:

$$\tilde{u}_n(x,t) = \sum_{k=0}^{n} \left(\frac{1}{\Gamma(k\beta+1)}\right) t^{k\beta}, 0 < \beta \le 1, x > 0, t > 0 \qquad (27)$$

Table 1. Absolute error of approximating the solution of Equation (25a) given that Equation (25b) to 5th order using FRDT method.

Variables		**Absolute error,** $E_5(u)=\lvert u_{exact}(x,t)-\tilde{u}_5(x,t)\rvert$			
t	x	$\alpha=0.25$	$\alpha=0.50$	$\alpha=0.75$	$\alpha=1$
0.25	0.25	4.852591	0.827357	0.530995	3.515836×10^{-7}
0.25	0.50	4.852591	0.827357	0.530995	3.515836×10^{-7}
0.25	0.75	4.852591	0.827357	0.530995	3.515836×10^{-7}
0.25	1.00	4.852591	0.827357	0.530995	3.515836×10^{-7}
0.50	0.25	6.317463	1.770632	0.449889	2.335403×10^{-5}
0.50	0.50	6.317463	1.770632	0.449889	2.335403×10^{-5}
0.50	0.75	6.317463	1.770632	0.449889	2.335403×10^{-5}
0.50	1.00	6.317463	1.770632	0.449889	2.335403×10^{-5}
0.75	0.25	7.356235	2.253027	0.544978	0.000276
0.75	0.50	7.356235	2.253027	0.544978	0.000276
0.75	0.75	7.356235	2.253027	0.544978	0.000276
0.75	1.00	7.356235	2.253027	0.544978	0.000276
1.00	0.25	8.108369	2.660339	0.591061	0.001615
1.00	0.50	8.108369	2.660339	0.591061	0.001615
1.00	0.75	8.108369	2.660339	0.591061	0.001615
1.00	1.00	8.108369	2.660339	0.591061	0.001615

By letting $n\in IN$ to ∞ or taking limit of both sides of Equation (27) as $n\in IN\to\infty$, the closed solution of Equation (25a) in the form of infinite fractional power series $u(x,t)$ is:

$$u(x,t)=\sum_{k=0}^{\infty}\frac{t^{k\beta}}{\Gamma(k\beta+1)}, 0<\beta\le 1,\ x>0,\ t>0 \tag{28}$$

Thus, by using Equation (6) in (28), the closed solution of Equation (25) in terms of Mittag-Leffler function one parameter is given by:

$$u(x,t)=E_\beta\left(t^\beta\right), 0<\beta\le 1,\ x>0,\ t>0 \tag{29}$$

If $\beta=\frac{1}{2}$, then Equation (29) becomes $u(x,t)=E_{\frac{1}{2}}\left(\sqrt{t}\right)$.

Lastly, the exact solution of Equation (28a), $u_{exact}(x,t)$ can be obtained from Equation (29) as β approaches 1 from left and is given by

$$u_{exact}(x,t)=e^t,\ \beta=1,\ x>0,\ t>0 \tag{30}$$

In order to demonstrate the agreement between the exact solution, Equation (30) and the n^{th} order approximate solution, Equation (27) of Equation (25a) given that Equation (25b), the absolute errors: $E_5(u)=\lvert u_{exact}(x,t)-\tilde{u}_5(x,t)\rvert$ and $E_6(u)=\lvert u_{exact}(x,t)-\tilde{u}_6(x,t)\rvert$ were computed as shown in Tables 1 and 2 by considering the 5^{th} order approximate solution, $\tilde{u}_5(x,t)=\sum_{k=0}^{5}\frac{t^{k\beta}}{\Gamma(k\beta+1)}$, $x,t,\beta\in\{0.25, 0.5, 0.75, 1\}$ and the 6^{th} order approximate solutions, $\tilde{u}_6(x,t)=\sum_{k=0}^{6}\frac{t^{k\beta}}{\Gamma(k\beta+1)}$, $x,t,\beta\in\{0.25, 0.5, 0.75, 1\}$ of Equation (25a) given that Equation (25b) without loss of generality.

Example 2: Taking $F(x)=-x,\ \lambda=1$ and choosing $f(x)=x$ in $(1.2a)$ (Kumar et al., 2012, 2017; Cetinkaya and Kiymaz, 2013), consider the initial value problem:

$$\begin{cases}\dfrac{\partial^\beta u}{\partial t^\beta}=\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial}{\partial x}(xu),\ x>0,\ \mathrm{t}>0,\ 0<\beta\le 1 & (31a)\\ \text{Subject to initial condition } u(x,0)=x & (31b)\end{cases}$$

Table 2. Absolute error of approximating the solution of Equation (25a) given that Equation (25b) to 6th order using FRDT method.

Variables		Absolute error, $E_6(u)=\lvert u_{exact}(x,t)-\tilde{u}_6(x,t)\rvert$			
t	x	$\alpha=0.25$	$\alpha=0.50$	$\alpha=0.75$	$\alpha=1$
0.25	0.25	4.915278	1.194657	0.047116	1.249937×10^{-8}
0.25	0.50	4.915278	1.194657	0.047116	1.249937×10^{-8}
0.25	0.75	4.915278	1.194657	0.047116	1.249937×10^{-8}
0.25	1.00	4.915278	1.194657	0.047116	1.249937×10^{-8}
0.50	0.25	6.494771	1.791466	0.087453	1.652645×10^{-6}
0.50	0.50	6.494771	1.791466	0.087453	1.652645×10^{-6}
0.50	0.75	6.494771	1.791466	0.087453	1.652645×10^{-6}
0.50	1.00	6.494771	1.791466	0.087453	1.652645×10^{-6}
0.75	0.25	7.681970	2.32334	0.126940	2.919142×10^{-5}
0.75	0.50	7.681970	2.32334	0.126940	2.919142×10^{-5}
0.75	0.75	7.681970	2.32334	0.126940	2.919142×10^{-5}
0.75	1.00	7.681970	2.32334	0.126940	2.919142×10^{-5}
1.00	0.25	8.609871	2.827005	0.595306	0.000226
1.00	0.50	8.609871	2.827005	0.595306	0.000226
1.00	0.75	8.609871	2.827005	0.595306	0.000226
1.00	1.00	8.609871	2.827005	0.595306	0.000226

Since $F(x)=-x, \lambda=1$ and $f(x)=x$, by Theorem 5, the FRDT of Equations (31a) and (31b) are:

$$\begin{cases} U_{k+1}(x)=\dfrac{\Gamma(k\beta+1)}{\Gamma(k\beta+\alpha+1)}\left(\dfrac{\partial^2}{\partial x^2}U_k(x)+U_k(x)+x\dfrac{\partial}{\partial x}U_k(x)\right), \quad 0<\beta\le 1,>0,\ x>0, k=0,1,2,\cdots & \text{(32a)} \\ U_0(x)=x, k=0 & \text{(32b)} \end{cases}$$

For $k=0, U_1(x)=\dfrac{2x}{\Gamma(\beta+1)}=\dfrac{2^1 x}{\Gamma(\beta+1)}, \quad 0<\beta\le 1, x>0, t>0$

For $k=1, U_2(x)=\dfrac{4x}{\Gamma(2\beta+1)}=\dfrac{2^2 x}{\Gamma(2\beta+1)}, \quad 0<\beta\le 1, x>0, t>0$

For $k=2, U_3(x)=\dfrac{8x}{\Gamma(3\beta+1)}=\dfrac{2^3 x}{\Gamma(3\beta+1)}, \quad 0<\beta\le 1, x>0, t>0$

Continuing with this procedure,

$$U_k(x)=\frac{2^k x}{\Gamma(k\beta+1)}, \ 0<\beta\le 1, x>0, t>0, \forall k=0,1,2,\cdots,n \ \& \ n\in IN \cdot$$

Then the inverse FRDT of $\{U_k(x)\}_{k=0}^{n}$ is the n^{th} order approximate solution of Equation (31a), $\tilde{u}_n(x,t)$ which is given by:

$$\tilde{u}_n(x,t)=\sum_{k=0}^{n} x\frac{2^k t^{k\beta}}{\Gamma(k\beta+1)}, 0<\beta\le 1, x>0, t>0 \qquad (33)$$

By letting $n\in IN$ to ∞ or taking limit of both sides of Equation (33) as $n\in IN\to\infty$, the closed solution of Equation (31a) in the form of infinite fractional power series $u(x,t)$ is:

$$u(x,t)=\sum_{k=0}^{\infty} x\frac{2^k t^{k\beta}}{\Gamma(k\beta+1)}, 0<\beta\le 1, x>0, t>0 \qquad (34)$$

Thus, by using Equation (6) in (34), the closed solution of Equation (31a) in terms of Mittag-Leffler function in one parameter is

$$u(x,t)=xE_\beta\left(2t^\beta\right), 0<\beta\le 1, x>0, t>0 \qquad (35)$$

If $\beta=\dfrac{1}{2}$, then Equation (35) becomes

Table 3. Absolute error of approximating the solution of Equation (31a) given that equation (31b) to 5th order using FRDT method.

Variable		Absolute error, $E_5(u) = \lvert u_{exact}(x,t) - \tilde{u}_5(x,t) \rvert$			
t	x	$\alpha = 0.25$	$\alpha = 0.50$	$\alpha = 0.75$	$\alpha = 1$
0.25	0.25	3.990596	0.932475	0.183504	5.838508×10^{-6}
0.25	0.50	7.981192	1.864950	0.367008	1.167702×10^{-5}
0.25	0.75	11.971788	2.797425	0.550512	1.751553×10^{-5}
0.25	1.00	15.962384	3.729899	0.734016	2.335403×10^{-5}
0.50	0.25	9.121892	1.893145	0.345244	0.000404
0.50	0.50	18.243785	3.786290	0.690488	0.000808
0.50	0.75	27.365677	5.679435	1.035732	0.001211
0.50	1.00	36.487569	7.572579	1.380976	0.001615
0.75	0.25	12.418329	3.102311	0.547420	0.004993
0.75	0.50	24.836659	6.204622	1.094840	0.009985
0.75	0.75	37.254988	9.306934	1.642259	0.014978
0.75	1.00	49.673318	12.409245	2.189679	0.019970
1.00	0.25	15.416232	4.497002	0.773588	0.030597
1.00	0.50	30.832463	8.994005	1.547175	0.061195
1.00	0.75	46.248695	13.491007	2.320763	0.091792
1.00	1.00	61.664927	17.988010	3.094351	0.122389

$$u(x,t) = xE_{\frac{1}{2}}\left(t^{\frac{1}{2}k}\right) = E_{\frac{1}{2}}\left(t^{\frac{k}{2}}\right)$$

Lastly, the exact solution of Equation (31a) $u_{exact}(x,t)$ is obtained from Equation (35) as β approaches 1 from left and it is given by

$$u_{exact}(x,t) = xe^{2t},\ \beta = 1,\ x > 0, t > 0 \tag{36}$$

In order to demonstrate the agreement between the exact solution, Equation 36 and the n^{th} order approximate solution, Equation (33) of (31a) given that Equation (31b), the absolute errors: $E_5(u) = \lvert u_{exact}(x,t) - \tilde{u}_5(x,t) \rvert$ and $E_6(u) = \lvert u_{exact}(x,t) - \tilde{u}_6(x,t) \rvert$ were computed as shown in Tables 3 and 4 by considering the 5^{th} order approximate solutions, $\tilde{u}_5(x,t) = \sum_{k=0}^{5} x \frac{2^k t^{k\beta}}{\Gamma(k\beta+1)}, x,t,\beta \in \{0.25, 0.5, 0.75, 1\}$ and the 6th order approximate solutions, $\tilde{u}_6(x,t) = \sum_{k=0}^{6} x \frac{2^k t^{k\beta}}{\Gamma(k\beta+1)}, x,t,\beta \in \{0.25, 0.5, 0.75, 1\}$ of Equation (31a) given that (31b) without loss of generality.

Example 3: Taking $F(x) = -x, \lambda = 1$ in (2a) and choosing $f(x) = x^2$ in Equation (2b) (Kumar et al., 2012, 2017; Cetinkaya and Kiymaz, 2013), consider the initial value problem:

$$\begin{cases} \dfrac{\partial^\beta u}{\partial t^\beta} = \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial}{\partial x}(xu), 0 < \beta \le 1, x > 0, t > 0 & \text{(37a)} \\ \text{Subject to the initial condition } : (x,0) = x^2 & \text{(37b)} \end{cases}$$

Since $F(x) = -x, \lambda = 1$ and $f(x) = x^2$, by Theorem 5, the FRDT of Equation (37a) and (37b) are:

$$\begin{cases} U_{k+1}(x) = \dfrac{\Gamma(k\beta+1)}{\Gamma(k\beta+\alpha+1)}\left(\dfrac{\partial^2}{\partial x^2}U_k(x) + U_k(x) + x\dfrac{\partial}{\partial x}U_k(x)\right),\ 0 < \beta \le 1, > 0,\ x > 0, k = 0, 1, 2, \cdots & \text{(38a)} \\ U_0(x) = x, k = 0 & \text{(38b)} \end{cases}$$

For $k = 0, U_1(x) = \dfrac{2+3x^2}{\Gamma(\beta+1)} = \dfrac{3^1(1+x^2)-1}{\Gamma(\beta+1)},\ 0 < \beta \le 1, x > 0, t > 0$

For $k = 1, U_2(x) = \dfrac{8+9x^2}{\Gamma(2\beta+1)} = \dfrac{3^2(1+x^2)-1}{\Gamma(2\beta+1)},\ 0 < \beta \le 1, x > 0, t > 0$

Table 4. Absolute error of approximating the solution of Equation (31a) given that equation (31b) to 6th order using FRDT method.

Variable		Absolute error, $E_6(u)=\lvert u_{exact}(x,t)-\tilde{u}_6(x,t)\rvert$			
t	x	$\alpha=0.25$	$\alpha=0.50$	$\alpha=0.75$	$\alpha=1$
0.25	0.25	6.553849	1.051051	0.083754	0.000549
0.25	0.50	13.107690	2.102100	0.167507	0.000917
0.25	0.75	19.661546	3.153153	0.251261	0.001376
0.25	1.00	26.215395	4.204204	0.335014	0.001834
0.50	0.25	11.958815	2.226478	0.348246	0.007387
0.50	0.50	23.917630	4.452956	0.696492	0.014774
0.50	0.75	35.876445	6.679435	1.044738	0.022160
0.50	1.00	47.835261	8.905913	1.392985	0.029547
0.75	0.25	17.630090	3.683119	0.566033	0.038147
0.75	0.50	35.260179	7.366238	1.132066	0.076294
0.75	0.75	52.890260	11.049357	1.698100	0.114441
0.75	1.00	70.520358	12.929421	2.264133	0.152588
1.00	0.25	23.440261	7.163669	0.841516	0.008375
1.00	0.50	46.880523	14.327338	1.683032	0.016750
1.00	0.75	70.320784	21.491007	2.524548	0.025125
1.00	1.00	93.761045	28.654676	3.366064	0.033501

For $k=2, U_3(x)=\dfrac{26+27x^2}{\Gamma(3\beta+1)}=\dfrac{3^3\left(1+x^2\right)-1}{\Gamma(3\beta+1)},\ \ 0<\beta\le 1, x>0, t>0$

Continuing with this procedure,

$$U_k(x)=\frac{3^k\left(1+x^2\right)-1}{\Gamma(k\beta+1)},\ 0<\beta\le 1, x>0, t>0, \forall k=0,1,2,\cdots,n\ \&\ n\in IN\cdot$$

Then the inverse FRDT of $\{U_k(x)\}_{k=0}^{n}$ denoted $\tilde{u}_n(x,t)$ is the n^{th} order approximate solution of Equation (37a) which is given by:

$$\tilde{u}_n(x,t)=\sum_{k=0}^{n}\frac{\left((1+x^2)-1\right)3^k t^{k\beta}}{\Gamma(k\beta+1)}, 0<\beta\le 1, x>0, t>0 \qquad (39)$$

By letting $n\in IN$ to ∞ or taking limit of both sides of Equation (39) as $n\in IN\to\infty$, the closed solution of Equation (37a) in the form of infinite fractional power series $u(x,t)$ is:

$$u(x,t)=\sum_{k=0}^{\infty}\frac{\left((1+x^2)-1\right)3^k t^{k\beta}}{\Gamma(k\beta+1)}, 0<\beta\le 1, x>0, t>0 \qquad (40)$$

Thus, by using Equation (6) in Equation (40), the closed solution of Equation (36a) in terms of Mittag-Leffler function is:

$$u(x,t)=x^2E_\beta\left(3t^\beta\right), 0<\beta\le 1, x>0, t>0 \qquad (41)$$

Lastly, the exact solution of Equation (37a), that is, $u_{exact}(x,t)$ which can be obtained from Equation (41) as β approaches 1 from left is

$$u_{exact}(x,t)=x^2e^{3t}, x>0, t>0 \qquad (42)$$

In order to demonstrate agreement between the exact solution, Equation (42) and the n^{th} order approximate solution, Equation (39) of (37a) given that (37b), the absolute errors: $E_5(u)=\lvert u_{exact}(x,t)-\tilde{u}_5(x,t)\rvert$ and $E_6(u)=\lvert u_{exact}(x,t)-\tilde{u}_6(x,t)\rvert$ were computed as shown in Tables 5 and 6 by considering the 5th order approximate solutions, $\tilde{u}_5(x,t)=\sum_{k=0}^{5}\frac{\left((1+x^2)-1\right)3^k t^{k\beta}}{\Gamma(k\beta+1)}, x,t,\beta\in\{0.25, 0.5, 0.75, 1\}$ and the 6th order approximate solutions, $\tilde{u}_6(x,t)=\sum_{k=0}^{6}\frac{\left((1+x^2)-1\right)3^k t^{k\beta}}{\Gamma(k\beta+1)}, x,t,\beta\in\{0.25, 0.5, 0.75, 1\}$

Table 5. Absolute error of approximating the solution of Equation (37a) given that (37b) to 5th order using FRDT method.

Variable		Absolute error, $E_6(u)=\lvert u_{exact}(x,t)-\tilde{u}_5(x,t)\rvert$			
t	x	$\alpha=0.25$	$\alpha=0.50$	$\alpha=0.75$	$\alpha=1$
0.25	0.25	5.176872	0.603496	0.057421	1.727399×10^{-5}
0.25	0.50	20.707489	2.413986	0.229682	1.909595×10^{-5}
0.25	0.75	46.591851	5.431468	0.516785	0.000155
0.25	1.00	82.829957	9.655943	0.918729	0.000276
0.50	0.25	9.871504	1.260829	0.139576	0.001248
0.50	0.50	39.486018	5.043315	0.558304	0.004993
0.50	0.75	88.84354	11.347458	1.256184	0.011233
0.50	1.00	157.944072	20.173259	2.233215	0.019970
0.75	0.25	14.593457	2.168811	0.464951	0.016227
0.75	0.50	58.373826	8.675243	1.859804	0.064909
0.75	0.75	131.341110	19.519297	4.184560	0.146045
0.75	1.00	233.495306	34.700973	7.439217	0.259635
1.00	0.25	19.138387	4.998305	0.743397 .	0.105346
1.00	0.50	76.553549	19.993219	2.973589	0.421384
1.00	0.75	172.245486	44.894743	6.690575	0.948115
1.00	1.00	306.214197	79.972876	11.894356	1.685537

Table 6. Absolute error of approximating the solution of Equation (37a) given that (37b) to 6th order using FRDT method.

Variable		Absolute error, $E_6(u)=\lvert u_{exact}(x,t)-\tilde{u}_6(x,t)\rvert$			
t	x	$\alpha=0.25$	$\alpha=0.50$	$\alpha=0.75$	$\alpha=1$
0.25	0.25	8.033082	0.722149	0.056544	1.824464×10^{-6}
0.25	0.50	32.132328	2.888595	0.226178	7.297854×10^{-6}
0.25	0.75	72.297739	6.499339	0.508899	1.642017×10^{-6}
0.25	1.00	128.529313	11.554381	0.904710	2.919142×10^{-6}
0.50	0.25	17.950086	2.210047	0.148125	0.000259
0.50	0.50	71.800343	8.840190	0.549992	0.001038
0.50	0.75	161.550771	19.890427	1.333122	0.002334
0.50	1.00	287.201370	35.360759	2.369995	0.004150
0.75	0.25	29.434758 .	5.372424	0.056544	0.004965
0.75	0.50	117.739032	21.489696	2.071823	0.019859
0.75	0.75	264.912821	48.351817	4.661602	0.044681
0.75	1.00	470.956127	85.958785	8.287292	0.079432
1.00	0.25	41.160618	12.592055	0.936834	0.042065
1.00	0.50	164.642474	50.368219	3.747335	0.168259
1.00	0.75	370.445566	113.328493	8.431503	0.378583
1.00	1.00	658.569895	201.478276	14.989338	0.673037

of Equation (37a) given that (37b) without loss of generality.

Example 4: Taking $F(x)=e^{-x}, \lambda=1$ in (2a) and choosing $f(x)=e^{x}$ in Equation (2b) (Kumar et al., 2012, 2017; Cetinkaya and Kiymaz, 2013), we have the initial value problem:

$$\begin{cases} \dfrac{\partial^{\beta}u}{\partial t^{\beta}}=\dfrac{\partial^{2}u}{\partial x^{2}}-\dfrac{\partial}{\partial x}(e^{-x}u), 0<\beta\leq 1, x>0, t>0 & (43a) \\ u(x,0)=e^{x} & (43b) \end{cases}$$

Since $F(x)=e^{-x}, \lambda=1$ and $f(x)=e^{x}$, by Theorem 5, the FRDT of Equation (43a) and (43b) are:

$$\begin{cases} U_{k+1}(x)=\dfrac{\Gamma(k\beta+1)}{\Gamma(k\beta+\alpha+1)}\left(\dfrac{\partial^{2}}{\partial x^{2}}U_{k}(x)+e^{-x}U_{k}(x)-e^{-x}\dfrac{\partial}{\partial x}U_{k}(x)\right), \\ 0<\beta\leq 1, >0,\ x>0, k=0, 1, 2,\cdots & (44a) \\ U_{0}(x)=e^{x}, k=0 & (44b) \end{cases}$$

For $k=0, U_{1}(x)=\dfrac{e^{x}}{\Gamma(\beta+1)},\ \ 0<\beta\leq 1,\ x>0, t>0$

For $k=1,\ U_{2}(x)=\dfrac{e^{x}}{\Gamma(2\beta+1)}, 0<\beta\leq 1,\ x>0, t>0$

For $k=2, U_{3}(x)=\dfrac{e^{x}}{\Gamma(3\beta+1)},\ 0<\beta\leq 1,\ x>0, t>0$

Continuing with this procedure,

$$U_{k}(x)=\frac{e^{x}}{\Gamma(k\beta+1)},\ 0<\beta\leq 1, x>0, t>0, \forall k=0, 1, 2,\cdots, n\ \&\ n\in IN\cdot$$

Then the inverse FRDT of $\{U_{k}(x)\}_{k=0}^{n}$ denoted by $u_{n}(x,t)$ is the n^{th} order approximate solution of (43a) which is given by:

$$\tilde{u}_{n}(x,t)=\sum_{k=0}^{n}\frac{e^{x}t^{k\beta}}{\Gamma(k\beta+1)}, 0<\beta\leq 1, x>0, t>0 \qquad (45)$$

By letting $n\in IN$ to ∞ or taking limit of both sides of equation (45) as $n\in IN\to\infty$, the closed form solution of Equation (43a) in infinite fractional power series, $u(x,t)$ is

$$u(x,t)=\sum_{k=0}^{\infty}\frac{e^{x}t^{k\beta}}{\Gamma(k\beta+1)}, 0<\beta\leq 1, x>0, t>0 \qquad (46)$$

Thus, by using Equation (6) in (46), the closed form solution of (43a) in terms of Mittag-Leffler function one parameter is given by:

$$u(x,t)=e^{x}E_{\beta}\left(t^{\beta}\right), 0<\beta\leq 1, x>0, t>0 \qquad (47)$$

Lastly, the exact solution of Equation (43a), that is, $u_{exact}(x,t)$ which can be obtained from Equation (47) as β approaches 1 from left is:

$$u_{exact}(x,t)=e^{x+t}, x>0, t>0 \qquad (48)$$

In order to demonstrate the agreement between the exact solution, Equation (47) and the nth order approximate solution, Equation (45) of (43a) given that (41b), the absolute errors: $E_{5}(u)=\left|u_{exact}(x,t)-\tilde{u}_{5}(x,t)\right|$ and $E_{6}(u)=\left|u_{exact}(x,t)-\tilde{u}_{6}(x,t)\right|$ were computed as shown in Tables 7 and 8 by considering the 5th order approximate solutions, $\tilde{u}_{5}(x,t)=\sum_{k=0}^{5}\frac{e^{x}t^{k\beta}}{\Gamma(k\beta+1)}, x,t,\beta\in\{0.25, 0.5, 0.75, 1\}$ and the 6th order approximate solutions, $\tilde{u}_{6}(x,t)=\sum_{k=0}^{6}\frac{e^{x}t^{k\beta}}{\Gamma(k\beta+1)}, x,t,\beta\in\{0.25, 0.5, 0.75, 1\}$ of Equation (43a) given that (43b) without loss of generality.

Example 5: Taking $F(x)=e^{-x+1}, \lambda=1$ in (2a) and choosing $f(x)=e^{x}$ in Equation (2b) consider the initial value problem:

$$\begin{cases} \dfrac{\partial^{\beta}u}{\partial t^{\beta}}=\dfrac{\partial^{2}u}{\partial x^{2}}-\dfrac{\partial}{\partial x}(e^{-x+1}u) & (49a) \\ u(x,0)=e^{x+1} & (49b) \end{cases}$$

Since $F(x)=e^{-x+1}, \lambda=1$ and $f(x)=e^{x+1}$, by Theorem 5, the FRDT of Equation (49a) and (49b) are:

$$\begin{cases} U_{k+1}(x)=\dfrac{\Gamma(k\beta+1)}{\Gamma(k\beta+\alpha+1)}\left(\dfrac{\partial^{2}}{\partial x^{2}}U_{k}(x)+e^{-x+1}U_{k}(x)-e^{-x+1}\dfrac{\partial}{\partial x}U_{k}(x)\right), \\ 0<\beta\leq 1, >0,\ x>0, k=0, 1, 2,\cdots & (50a) \\ U_{0}(x)=e^{x+1}, k=0 & (50b) \end{cases}$$

For $k=0,\ U_{1}(x)=\dfrac{e^{x+1}}{\Gamma(\beta+1)},\ 0<\beta\leq 1,\ x>0, t>0$

Table 7. Absolute error of approximating the solution of Equation (43a) given that (43b) to 5th order using FRDT method.

Variable		Absolute error, $E_5(u)=\lvert u_{exact}(x,t)-\tilde{u}_5(x,t)\rvert$			
t	x	$\alpha=0.25$	$\alpha=0.50$	$\alpha=0.75$	$\alpha=1$
0.25	0.25	6.230850	1.062347	0.681811	4.514423×10^{-7}
0.25	0.50	8.000570	1.364081	0.875463	5.796634×10^{-7}
0.25	0.75	10.272935	1.751515	1.124116	7.443025×10^{-7}
0.25	1.00	13.190710	2.248989	1.443394	9.557033×10^{-7}
0.50	0.25	8.617463	2.273536	0.577669	2.998717×10^{-5}
0.50	0.50	10.910352	2.919279	0.741742	3.850429×10^{-5}
0.50	0.75	14.009169	3.748428	0.952415	4.944048×10^{-5}
0.50	1.00	17.988129	4.813077	1.222925	6.348284×10^{-5}
0.75	0.25	9.445593	2.892944	0.699766	0.000354
0.75	0.50	12.128381	3.714614	0.898517	0.000455
0.75	0.75	15.573150	4.769658	1.153718	0.000584
0.75	1.00	19.996320	6.124362	1.481404	0.000750
1.00	0.25	10.108369	3.415943	0.758937	0.002074
1.00	0.50	13.368440	4.386157	0.974495	0.002663
1.00	0.75	17.165417	5.631938	1.251276	0.003419
1.00	1.00	22.040832	7.231551	1.606670	0.004390

Table 8. Absolute error of approximating the solution of Equation (43a) given that (43a) to 6th order using FRDT method.

Variable		Absolute error, $E_6(u)=\lvert u_{exact}(x,t)-\tilde{u}_6(x,t)\rvert$			
t	x	$\alpha=0.25$	$\alpha=0.50$	$\alpha=0.75$	$\alpha=1$
0.25	0.25	6.915278	1.533970	0.060498	1.604951×10^{-8}
0.25	0.50	8.103923	1.969656	0.077681	2.060798×10^{-8}
0.25	0.75	10.405644	2.529089	0.099745	2.646117×10^{-8}
0.25	1.00	13.361111	3.388529	0.128075	3.397681×10^{-8}
0.50	0.25	8.339451	2.300288	0.112292	2.122038×10^{-6}
0.50	0.50	10.708067	2.953628	0.144186	2.724751×10^{-6}
0.50	0.75	13.749430	3.792534	0.185138	3.498649×10^{-6}
0.50	1.00	17.654618	4.869709	0.237722	4.492355×10^{-6}
0.75	0.25	9.863845	2.983228	0.162994	3.748253×10^{-5}
0.75	0.50	12.665427	3.830540	0.209289	4.812852×10^{-5}
0.75	0.75	16.262731	4.918511	0.268732	6.179824×10^{-5}
0.75	1.00	20.712492	6.315493	0.345059	7.935051×10^{-5}
1.00	0.25	11.055293	3.629946	0.764388	0.000290
1.00	0.50	14.195277	4.660943	0.981494	0.000373
1.00	0.75	18.227097	5.984770	1.260263	0.000478
1.00	1.00	23.404056	7.684596	1.618209	0.000614

Table 9. Absolute error of approximating the solution of Equation (49a) given that (49b) to 5th order using FRDTM.

Variable		Absolute error, $E_5(u)=\lvert u_{exact}(x,t)-\tilde{u}_5(x,t)\rvert$			
t	x	$\alpha=0.25$	$\alpha=0.50$	$\alpha=0.75$	$\alpha=1$
0.25	0.25	16.937206	2.887759	1.853354	1.227147 $E-6$
0.25	0.50	21.747804	3.707957	2.379755	1.575688 $E-6$
0.25	0.75	27.924733	4.761111	3.055664	2.022490 $E-6$
0.25	1.00	35.856067	6.248989	3.923552	2.597871 $E-6$
0.50	0.25	23.424693	6.180112	1.570267	8.151358 $E-5$
0.50	0.50	29.657412	7.935423	2.016264	0.000105
0.50	0.75	38.080887	10.189284	2.588932	0.000134
0.50	1.00	48.896804	13.083300	3.324255	0.000173
0.75	0.25	25.675784	7.863837	1.902161	0.000962
0.75	0.50	32.968358	10.097368	2.442422	0.001237
0.75	0.75	42.332211	12.965275	3.136131	0.001587
0.75	1.00	54.355633	16.647742	4.026874	0.002039
1.00	0.25	27.477396	9.285496	2.063005	0.005638
1.00	0.50	36.339188	11.922811	2.648952	0.007239
1.00	0.75	46.660441	15.309195	3.401321	0.009294
1.00	1.00	59.911801	19.657394	4.367382	0.011933

For $k=1,\ U_2(x)=\dfrac{e^{x+1}}{\Gamma(2\beta+1)},\ 0<\beta\le 1,\ x>0, t>0$

For $k=2, U_3(x)=\dfrac{e^{x+1}}{\Gamma(3\beta+1)},\ 0<\beta\le 1,\ x>0, t>0$

Continuing with this procedure,

$$U_k(x)=\frac{e^{x+1}}{\Gamma(k\beta+1)},\ 0<\beta\le 1, x>0, t>0, \forall k=k=0, 1, 2,\cdots,n.\ \&\ n\in IN.$$

Then the inverse FRDT of $\{U_k(x)\}_{k=0}^{n}$ denoted $\tilde{u}_n(x,t)$ is the n^{th} order approximate solution of Equation (49a) which is given by:

$$\tilde{u}_n(x,t)=\sum_{k=0}^{n}\frac{e^{x+1}t^{k\beta}}{\Gamma(k\beta+1)},0<\beta\le 1, x>0, t>0 \qquad (51)$$

By letting $n\in IN$ to ∞ or taking limit of both sides of Equation (51) as $n\in IN\to\infty$, the closed solution of Equation (49a) in the form of infinite fractional power series, $u(x,t)$ is

$$u(x,t)=\sum_{k=0}^{\infty}\frac{e^{x+1}t^{k\beta}}{\Gamma(k\beta+1)},0<\beta\le 1, x>0, t>0 \qquad (52)$$

Thus, by using Equation (6) in (52), the closed form solution of Equation (49a) in terms of Mittag-Leffler function is given by:

$$u(x,t)=e^{x+1}E_\beta\left(t^\beta\right), 0<\beta\le 1, x>0, t>0 \qquad (53)$$

Lastly, the exact solution of Equation (49a) which can be obtained from Equation (53) as β approaches 1 from left is:

$$u_{exact}(x,t)=e^{x+t+1},\ x>0, t>0 \qquad (54)$$

In order to demonstrate the agreement between the exact solution, Equation (54) and the nth order approximate solution, Equation (51) of (49a) given that (49b), the absolute errors: $E_5(u)=\lvert u_{exact}(x,t)-\tilde{u}_5(x,t)\rvert$ and $E_6(u)=\lvert u_{exact}(x,t)-\tilde{u}_6(x,t)\rvert$ were computed as shown in Tables 9 and 10 by considering the 5th order approximate solutions, $\tilde{u}_5(x,t)=\sum_{k=0}^{5}\frac{e^{x+1}t^{k\beta}}{\Gamma(k\beta+1)}, x,t,\beta\in\{0.25, 0.5, 0.75, 1\}$ and the 6th order approximate solutions, $\tilde{u}_6(x,t)=\sum_{k=0}^{6}\frac{e^{x+1}t^{k\beta}}{\Gamma(k\beta+1)}, x,t,\beta\in\{0.25, 0.5, 0.75, 1\}$ of

Table 10. Absolute error of approximating the solution of Equation (49a) given that (49b) to 6th order using FRDTM.

Variables		**Absolute error,** $E_6(u)=\left\|u_{exact}(x,t)-\tilde{u}_6(x,t)\right\|$			
t	x	$\alpha=0.25$	$\alpha=0.50$	$\alpha=0.75$	$\alpha=1$
0.25	0.25	18.797675	4.169763	0.164451	4.362709 $E-8$
0.25	0.50	22.028747	4.354080	0.211159	5.601830 $E-8$
0.25	0.75	28.285473	6.874777	0.271135	7.192892 $E-8$
0.25	1.00	36.319265	9.210977	0.348144	9.235855 $E-8$
0.50	0.25	22.668978	6.252831	0.305241	5.768297 $E-6$
0.50	0.50	29.107544	8.028793	0.391938	7.406641 $E-6$
0.50	0.75	37.374826	10.309176	0.503257	9.510314 $E-6$
0.50	1.00	47.990227	13.237241	0.646195	1.221149 $E-5$
0.75	0.25	26.812711	8.109254	0.443064	0.000102
0.75	0.50	34.428200	10.412487	0.568906	0.000131
0.75	0.75	44.206686	13.369899	0.730489	0.000168
0.75	1.00	56.302391	17.167290	0.937968	0.000216
1.00	0.25	30.051402	9.867216	2.077822	0.000788
1.00	0.50	38.586764	12.669757	2.667977	0.001014
1.00	0.75	49.546387	16.268292	3.425750	0.001299
1.00	1.00	63.618820	20.888898	4.398748	0.001669

Equation (49a) given that (49b) without loss of generality.

DISCUSSION

Here, the results obtained from the five application examples considered above are discussed. Through the first three aforementioned examples, the fractional reduced differential transform method (FRDTM) was successfully applied to the time fractional diffusion equations, that is, Equation (2a) given that (2b), for $F(x)=-x, \lambda=1$, the different initial conditions $f(x)=1$, $f(x)=x$ and $f(x)=x^2$, and $0<\beta\leq 1$. As a result, the closed solutions of Equation (2a) given that (2b) in the form of infinite fractional power series and in terms of Mittag-Leffler function in one parameter as well as its exact solutions were obtained and are in complete agreement with the results obtained by Cetinkaya and Kiymaz (2013), Kumar et al. (2012) , Kumar et al. (2017). For $\beta=\frac{1}{2}$ with the same $F(x)$, λ and $f(x)$ specified above, the closed solutions of Equation (2a) given that (2b) in the form of infinite fractional power series and in terms of Mittag-Leffler function in one parameter as well as their exact solutions, which were obtained by FRDTM, are in complete agreement with the results obtained by Das (2009).

Through the fourth example above, FRDTM was applied to Equation (2a) given that (2b), where $F(x)=e^{-x}$, $\lambda=1$, $f(x)=e^{x}$ and $0<\beta\leq 1$, and the closed solutions in the form of infinite fractional power series and in terms of Mittag-Leffler function in one parameter as well as exact solution were obtained. The results obtained here are in complete agreement with the results obtained by Cetinkaya and Kiymaz (2013).

Through the fifth example mentioned above, FRDTM was applied to Equation (2a) given that (2b), where $F(x)=e^{-x+1}$, $\lambda=1$, $f(x)=e^{x+1}$ and $0<\beta\leq 1$, and the closed solutions in the form of infinite fractional power series and in terms of Mittag-Leffler function in one parameter as well as exact solution were obtained.

Without loss of generality, only the 5th and 6th order approximate solutions of Equations (25a), (31a), (34a), (43a) and (49a); $\forall\beta\in\{0.25, 0.5, 0.75, 1\}\subseteq(0,1]$ and $\forall(x,t)\in\{0.25, 0.5, 0.75, 1\}\times\{0.25, 0.5, 0.75, 1\}\subseteq[0,1]\times[0,1]$ were considered to compute the absolute errors in this paper. The validity, accuracy and convergence of the FRDTM was checked through the computed absolute errors, $E_5(u)=\left|u_{exact}(x,t)-\tilde{u}_5(x,t)\right|$ and $E_6(u)=\left|u_{exact}(x,t)-\tilde{u}_6(x,t)\right|$ for each values of parameter

$\beta \in \{0.25, 0.5, 0.75, 1\} \subseteq (0,1]$, where $u(x,t)$ is the exact solutions, $u_5(x,t)$ is the 5th order approximate solutions and $u_6(x,t)$ is the 6th order approximate solutions of each of the five examples considered earlier. From observation made through Tables 1 to 10, the absolute errors: $E_5(u)$ and $E_6(u)$ decrease as $\beta \in \{0.25, 0.5, 0.75, 1\}$ increases from 0.25 to 1. This implies that the 5th and 6th order approximate solutions of Equations (25a), (31a), (34a), (43a) and (49a) converge to their exact solutions as $\beta \in \{0.25, 0.5, 0.75, 1\}$ increases from 0.25 to 1. It was also observed that for each $(x,t) \in \{0.25, 0.5, 0.75, 1\} \times \{0.25, 0.5, 0.75, 1\}$ and for each $\beta \in \{0.25, 0.5, 0.75, 1\}$ throughout Tables 1 to 10, $E_5(u) > E_6(u)$. This shows that the validity, accuracy and convergence of the fractional power series solutions of Equations (25a), (31a), (34a), (43a) and (49a) can be improved by calculating more term in the series solutions by using the present method (FRDTM).

Conclusion

In this study, a new general recurrence relation for fractional reduced transform of time fractional diffusion equations with initial conditions of the form (2a) given that (2b) was developed and then methodology of fractional reduced differential transform method (FRDTM) was also developed with the help of this new general recurrence relation. The fractional reduced differential transform method was applied to five time fractional diffusion equations with initial conditions, which exist in the literature except the last one, to obtain their closed solutions in the form of infinite fractional power series and in terms of Mittag-Leffler function in one parameter as well as exact solutions. The results evaluated for the first four time fractional diffusion equations are in good agreement with the ones already existing in the literature. Precisely, the general recurrence relation works successfully in solving time fractional diffusion equations with initial conditions by using fractional reduced differential transform method to obtain their closed solutions in the form of infinite fractional power series and in terms of Mittag-Leffler function one parameter as well as exact solutions with a minimum size of calculations.

Thus, it can be said that the general recurrence relation used in solving time-fractional diffusion equations with initial conditions by using fractional reduced differential transform method can be extended to solve other fractional partial differential equations with initial conditions which can arise in fields of sciences.

CONFLICT OF INTERESTS

The author has not declared any conflict of interests.

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