Engineering and Applied Science Letters

PSR Press

*Article*

# A new forms of extended hypergeometric functions and their properties

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Received: 31 March 2020; Accepted: 31 January 2021; Published: 25 February 2021.

**Abstract:** In this paper, a new forms of extended hypergeometric functions are introduced. Some functional relations, integral representations and transformation formulas for these functions are derived.

**Keywords:** Gamma function, Beta function, Gauss hypergeometric function, confluent hypergeometric function, integral representations.

## 1. Introduction

In recent years, some extensions of special functions have been considered by many authors [1–8]. In 1994, Chaudhry and Zubair [3] have introduced the following extension of Gamma function

$$\Gamma_p(x) = \int_0^\infty t^{x-1} \exp(-t - pt^{-1})\, dt; \qquad (Re(p) \geq 0)\,. \tag{1}$$

In 1997, Chaudhry *et al.*, [1] presented the following extension of Euler's Beta function

$$B_p(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} \exp\left[\frac{-p}{t(1-t)}\right] dt, \qquad (Re(p) \geq 0;\ Re(x) > 0;\ Re(y) > 0)\,. \tag{2}$$

It is clearly seem that $\Gamma_0(x) = \Gamma(x)$ and $B_0(x,y) = B(x,y)$, where $\Gamma(x)$ and $B(x,y)$ are the classical Gamma and Beta functions (see [9]).

Afterwards, Chaudhry *et al.*, [2] used $B_p(x,y)$ to defined the Gauss hypergeometric and confluent hypergeometric functions as

$$F_p(a,b;c;z) = \sum_{n=0}^{\infty} (a)_n \frac{B_p(b+n, c-b)}{B(b,c-b)} \frac{z^n}{n!}, \qquad (Re(p) \geq 0;\ |z| < 1;\ Re(c) > Re(b) > 0)\,,$$

$$\Phi_p(b;c;z) = \sum_{n=0}^{\infty} \frac{B_p(b+n, c-b)}{B(b,c-b)} \frac{z^n}{n!}, \qquad (Re(p) \geq 0;\ Re(c) > Re(b) > 0)\,,$$

respectively, where $(a)_n$ denotes the Pchhammer symbol [9] defined as $(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}$ and $(a)_0 = 1$. Also, $F_p(a,b;c;z)$ and $\Phi_p(b;c;z)$ are defined as [2]

$$F_p(a,b;c;z) = \frac{1}{B(b,c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a} \exp\left[\frac{-p}{t(1-t)}\right] dt, \tag{3}$$

$$(Re(p) \geq 0;\ |\arg(1-z)| < \pi;\ Re(c) > Re(b) > 0)\,,$$

$$\Phi_p(b;c;z) = \frac{1}{B(b,c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1} e^{zt} \exp\left[\frac{-p}{t(1-t)}\right] dt, \qquad (Re(p) \geq 0;\ Re(c) > Re(b) > 0)\,.$$

Note that

$$F_0(a,b;c;z) = {}_2F_1(a,b;c;z),$$
$$\Phi_0(b;c;z) = \Phi(b;c;z) = {}_1F_1(b;c;z).$$

In 2011, Özergin [10] (see also Özergin *et al.* [6]) introduced a further extension of the Gamma and Beta functions as

$$\Gamma_p^{(\alpha,\beta)}(x) = \int_0^\infty t^{x-1}{}_1F_1\left(\alpha;\beta;-t-\frac{p}{t}\right)dt, \qquad (Re(p)\geq 0;\ Re(\alpha)>0;\ Re(\beta)>0;\ Re(x)>0), \tag{4}$$

and

$$B_p^{(\alpha,\beta)}(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}{}_1F_1\left(\alpha;\beta;\frac{-p}{t(1-t)}\right)dt, \tag{5}$$

$$(Re(p)\geq 0;\ Re(\alpha)>0;\ Re(\beta)>0;\ Re(x)>0;\ Re(y)>0),$$

respectively. Clearly, we have $\Gamma_p^{(\alpha,\alpha)}(x) = \Gamma_p(x)$, $\Gamma_0^{(\alpha,\alpha)}(x) = \Gamma(x)$, $B_p^{(\alpha,\alpha)}(x,y) = B_p(x,y)$ and $B_0^{(\alpha,\beta)}(x,y) = B(x,y)$. In [6], $B_p^{(\alpha,\beta)}(x,y)$ is used to defined $F_p^{(\alpha,\beta)}(a,b;c;z)$ and $\Phi_p^{(\alpha,\beta)}(b;c;z)$ as

$$F_p^{(\alpha,\beta)}(a,b;c;z) = \sum_{n=0}^\infty (a)_n \frac{B_p^{(\alpha,\beta)}(b+n,c-b)}{B(b,c-b)}\frac{z^n}{n!},$$

$$(Re(p)\geq 0;\ |z|<1;\ Re(c)>Re(b)>0;\ Re(\alpha)>0;\ Re(\beta)>0),$$

and

$$\Phi_p^{(\alpha,\beta)}(b;c;z) = \sum_{n=0}^\infty \frac{B_p^{(\alpha,\beta)}(b+n,c-b)}{B(b,c-b)}\frac{z^n}{n!}, \quad (Re(p)\geq 0;\ Re(c)>Re(b)>0;\ Re(\alpha)>0;\ Re(\beta)>0),$$

respectively. We observe that

$$F_p^{(\beta,\beta)}(a,b;c;z) = F_p(a,b;c;z),$$
$$F_0^{(\alpha,\beta)}(a,b;c;z) = {}_2F_1(a,b;c;z),$$

and

$$\Phi_p^{(\beta,\beta)}(b;c;z) = \Phi_p(b;c;z),$$
$$\Phi_0^{(\alpha,\beta)}(b;c;z) = \Phi(b;c;z) = {}_1F_1(b;c;z).$$

Very recently, Shadab *et al.*, [11] introduced another form of the extended Beta function as

$$B_\alpha^p(x,y) = \int_0^1 t^{t-1}(1-t)^{y-1}E_\alpha\left(\frac{-p}{t(1-t)}\right)dt, \quad (\alpha\in R_0^+;\ Re(x)>0;\ Re(y)>0;\ Re(p)\geq 0), \tag{6}$$

where $E_\alpha(z)$ is the classical Mittag-Leffler function given in [12].

The above extended Beta function (6) is used to defined $F_{p,\alpha}(a,b;c;z)$ and $\Phi_{p,\alpha}(b;c;z)$ as [11]

$$F_{p,\alpha}(a,b;c;z) = \sum_{n=0}^\infty (a)_n \frac{B_\alpha^p(b+n,c-b)}{B(b,c-b)}\frac{z^n}{n!}, \qquad (\alpha\in R^+;\ p\in R_0^+;\ |z|<1;\ Re(c)>Re(b)>0),$$

and

$$\Phi_{p,\alpha}(b;c;z) = \sum_{n=0}^\infty \frac{B_\alpha^p(b+n,c-b)}{B(b,c-b)}\frac{z^n}{n!}, \qquad (\alpha\in R^+;\ p\in R_0^+;\ Re(c)>Re(b)>0).$$

Also, very recently, Al-Gonah and Mohammed [13], extended the Gamma and Beta functions as [13]

$$\Gamma_p^{(\alpha,\beta,\gamma)}(x) = \int_0^\infty t^{x-1} E_{\alpha,\beta}^{\gamma}\left(-t-\frac{p}{t}\right) dt, \quad (Re(p) \geq 0;\ Re(\alpha) > 0;\ Re(\beta) > 0;\ Re(\gamma) > 0, Re(x) > 0), \tag{7}$$

$$B_p^{(\alpha,\beta,\gamma)}(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} E_{\alpha,\beta}^{\gamma}\left(\frac{-p}{t(1-t)}\right) dt, \quad (Re(p) \geq 0;\ Re(\alpha) > 0;\ Re(\beta) > 0;\ Re(\gamma) > 0), \tag{8}$$

where $E_{\alpha,\beta}^{\gamma}(z)$ denotes the generalized Mittag-Leffler function defined as [14]

$$E_{\alpha,\beta}^{\gamma}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_n}{\Gamma(\alpha n+\beta)} \frac{z^n}{n!}, \qquad (z,\alpha,\beta,\gamma \in \mathbb{C};\ Re(\alpha) > 0;\ Re(\beta) > 0;\ Re(\gamma) > 0).$$

It clear that

$$\begin{aligned} \Gamma(\beta) E_{1,\beta}^{\gamma}(z) &= {}_1F_1(\gamma;\beta;z), \\ E_{\alpha,1}^{1}(z) &= E_\alpha(z), \\ E_{1,1}^{1}(z) &= e^z. \end{aligned}$$

From (7) and (8), we note that

$$\begin{aligned} \Gamma_p^{(1,\beta,\gamma)}(x) &= \frac{1}{\Gamma(\beta)} \Gamma_p^{(\gamma,\beta)}(x), \\ B_p^{(1,\beta,\gamma)}(x,y) &= \frac{1}{\Gamma(\beta)} B_p^{(\gamma,\beta)}(x,y), \\ \Gamma_p^{(1,1,1)}(x) &= \Gamma_p(x), \\ B_p^{(1,1,1)}(x,y) &= B_p(x,y), \end{aligned}$$

where $\Gamma_p^{(\gamma,\beta)}(x)$, $B_p^{(\gamma,\beta)}(x,y)$, $\Gamma_p(x)$ and $B_p(x,y)$ denote the various forms of generalized Gamma and Beta functions given in (4), (5), (1) and (2) respectively. Also, we not that

$$B_p^{(\alpha,1,1)}(x,y) = B_\alpha^p(x,y),$$

where $B_\alpha^p(x,y)$ denotes the new extended Beta function given in (6).

## 2. A new forms of hypergeometric functions

In this section, we use the new extended Beta function (8) to introduce a new forms of extended Gauss hypergeometric and confluent hypergeometric functions as follows:

$$F_p^{(\alpha,\beta,\gamma)}(a,b;c;z) = \sum_{n=0}^{\infty} (a)_n \frac{B_p^{(\alpha,\beta,\gamma)}(b+n,c-b)}{B(b,c-b)} \frac{z^n}{n!}, \tag{9}$$

$$(Re(p) \geq 0;\ |z| < 1;\ Re(c) > Re(b) > 0;\ Re(\alpha) > 0;\ Re(\beta) > 0;\ Re(\gamma) > 0),$$

and

$$\Phi_p^{(\alpha,\beta,\gamma)}(b;c;z) = \sum_{n=0}^{\infty} \frac{B_p^{(\alpha,\beta,\gamma)}(b+n,c-b)}{B(b,c-b)} \frac{z^n}{n!}, \tag{10}$$

$$(Re(p) \geq 0;\ Re(c) > Re(b) > 0;\ Re(\alpha) > 0;\ Re(\beta) > 0;\ Re(\gamma) > 0),$$

respectively. Also, we observe that

$$F_p^{(1,\beta,\gamma)}(a,b;c;z) = \frac{1}{\Gamma(\beta)} F_p^{(\gamma,\beta)}(a,b;c;z), \tag{11}$$

$$F_p^{(\alpha,1,1)}(a,b;c;z) = F_{p,\alpha}(a,b;c;z), \tag{12}$$

$$F_p^{(1,1,1)}(a,b;c;z) = F_p(a,b;c;z), \tag{13}$$

$$F_0^{(\alpha,1,\gamma)}(a,b;c;z) = {}_2F_1(a,b;c;z), \tag{14}$$

and

$$\Phi_p^{(1,\beta,\gamma)}(b;c;z) = \frac{1}{\Gamma(\beta)} \Phi_p^{(\gamma,\beta)}(b;c;z), \tag{15}$$

$$\Phi_p^{(\alpha,1,1)}(b;c;z) = \Phi_{p,\alpha}(b;c;z), \tag{16}$$

$$\Phi_p^{(1,1,1)}(b;c;z) = \Phi_p(b;c;z), \tag{17}$$

$$\Phi_0^{(\alpha,1,\gamma)}(b;c;z) = {}_1F_1(b;c;z) = \Phi(b;c;z). \tag{18}$$

Some properties of the above functions are given in the form of the following theorems:

**Theorem 1.** *For $F_p^{(\alpha,\beta,\gamma)}(a,b;c;z)$ and $\Phi_p^{(\alpha,\beta,\gamma)}(b;c;z)$, the following functional relations hold;*

$$c\, F_p^{(\alpha,\beta,\gamma)}(a,b;c;z) = b\, F_p^{(\alpha,\beta,\gamma)}(a,b+1;c+1;z) + (c-b) F_p^{(\alpha,\beta,\gamma)}(a,b;c+1;z), \tag{19}$$

$$c\, \Phi_p^{(\alpha,\beta,\gamma)}(b;c;z) = b\, \Phi_p^{(\alpha,\beta,\gamma)}(b+1;c+1;z) + (c-b) \Phi_p^{(\alpha,\beta,\gamma)}(b;c+1;z). \tag{20}$$

**Proof.** Using the following known relation [13]

$$B_p^{(\alpha,\beta,\gamma)}(x,y) = B_p^{(\alpha,\beta,\gamma)}(x+1,y) + B_p^{(\alpha,\beta,\gamma)}(x,y+1), \tag{21}$$

in (9), we get

$$\begin{aligned} F_p^{(\alpha,\beta,\gamma)}(a,b;c;z) &= \sum_{n=0}^{\infty} (a)_n \left[ \frac{B_p^{(\alpha,\beta,\gamma)}(b+n+1,c-b) + B_p^{(\alpha,\beta,\gamma)}(b+n,c-b+1)}{B(b,c-b)} \right] \frac{z^n}{n!}, \\ &= \sum_{n=0}^{\infty} (a)_n \frac{B_p^{(\alpha,\beta,\gamma)}(b+n+1,c-b)}{B(b,c-b)} \frac{z^n}{n!} + \sum_{n=0}^{\infty} (a)_n \frac{B_p^{(\alpha,\beta,\gamma)}(b+n,c-b+1)}{B(b,c-b)} \frac{z^n}{n!}, \\ &= \frac{B(b+1,c-b)}{B(b,c-b)} \sum_{n=0}^{\infty} (a)_n \frac{B_p^{(\alpha,\beta,\gamma)}(b+n+1,c-b)}{B(b+1,c-b)} \frac{z^n}{n!} \\ &\quad + \frac{B(b,c-b+1)}{B(b,c-b)} \sum_{n=0}^{\infty} (a)_n \frac{B_p^{(\alpha,\beta,\gamma)}(b+n,c-b+1)}{B(b,c-b+1)} \frac{z^n}{n!}, \end{aligned}$$

which on using the relation [9] $B(x,y) = \frac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)}$, and (9) yields the desired result.

Similarly, using (21) in (10) and following the same procedure leading to result (19), we get the desired result (20) and thus the proof of Theorem 1 is completed. □

Using [13]

$$(1+\alpha\,\gamma-\beta)\, B_p^{(\alpha,\beta,\gamma)}(x,y) = \alpha\,\gamma\, B_p^{(\alpha,\beta,\gamma+1)}(x,y) - B_p^{(\alpha,\beta-1,\gamma)}(x,y),$$

$$p\, B_p^{(\alpha,\beta,\gamma)}(x-1,y-1) = B_p^{(\alpha,\beta-\alpha,\gamma-1)}(x,y) - B_p^{(\alpha,\beta-\alpha,\gamma)}(x,y),$$

together in (9) and (10) respectively and proceeding on the same lines of proof of Theorem 1, we get the following functional relations:

**Theorem 2.** *For $F_p^{(\alpha,\beta,\gamma)}(a,b;c;z)$ and $\Phi_p^{(\alpha,\beta,\gamma)}(b;c;z)$, the following functional relations hold;*

$$(1+\alpha\,\gamma-\beta)\,F_p^{(\alpha,\beta,\gamma)}(a,b;c;z) = \alpha\,\gamma\,F_p^{(\alpha,\beta,\gamma+1)}(a,b;c;z) - F_p^{(\alpha,\beta-1,\gamma)}(a,b;c;z),$$
$$(1+\alpha\,\gamma-\beta)\,\Phi_p^{(\alpha,\beta,\gamma)}(b;c;z) = \alpha\,\gamma\,\Phi_p^{(\alpha,\beta,\gamma+1)}(b;c;z) - \Phi_p^{(\alpha,\beta-1,\gamma)}(b;c;z),$$
$$(\alpha,\beta,\gamma\in\mathbb{C};\ Re(\alpha)>0;\ Re(\gamma)>0;\ Re(\beta)>1),$$

$$p\,c\,(c+1)\,F_p^{(\alpha,\beta,\gamma)}(a,b;c;z) = b\,(c-b)\left\{F_p^{(\alpha,\beta-\alpha,\gamma-1)}(a,b+1;c+2;z) - F_p^{(\alpha,\beta-\alpha,\gamma)}(a,b+1;c+2;z)\right\},$$
$$p\,c\,(c+1)\Phi_p^{(\alpha,\beta,\gamma)}(b;c;z) = b\,(c-b)\left\{\Phi_p^{(\alpha,\beta-\alpha,\gamma-1)}(b+1;c+2;z) - \Phi_p^{(\alpha,\beta-\alpha,\gamma)}(b+1;c+2;z)\right\},$$
$$(Re(p)\geq 0;\ Re(c)>Re(b)>0;\ Re(\beta)>Re(\alpha)>0;\ Re(\gamma)>1).$$

Using [13]

$$B_p^{(\alpha,\beta,\gamma)}(x,y) = \sum_{n=0}^{\infty} B_p^{(\alpha,\beta,\gamma)}(x+n,y+1),$$
$$B_p^{(\alpha,\beta,\gamma)}(x,1-y) = \sum_{n=0}^{\infty} \frac{(y)_n}{n!} B_p^{(\alpha,\beta,\gamma)}(x+n,1),$$
$$B_p^{(\alpha,\beta,\gamma)}(x,y) = \sum_{n=0}^{k} \binom{k}{n} B_p^{(\alpha,\beta,\gamma)}(x+n,y+k-n), \quad (k\in\mathbb{N}),$$

together in (9) and (10) respectively and proceeding on the same lines of proof of Theorem 1, we get the following summation relations:

**Theorem 3.** *For $F_p^{(\alpha,\beta,\gamma)}(a,b;c;z)$ and $\Phi_p^{(\alpha,\beta,\gamma)}(b;c;z)$, the following summation relations hold;*

$$F_p^{(\alpha,\beta,\gamma)}(a,b;c;z) = (c-b)\sum_{k=0}^{\infty}\frac{(b)_k}{(c)_{k+1}} F_p^{(\alpha,\beta,\gamma)}(a,b+k;c+k+1;z),$$
$$\Phi_p^{(\alpha,\beta,\gamma)}(b;c;z) = (c-b)\sum_{k=0}^{\infty}\frac{(b)_k}{(c)_{k+1}} \Phi_p^{(\alpha,\beta,\gamma)}(b+k;c+k+1;z),$$
$$F_p^{(\alpha,\beta,\gamma)}(a,b;c;z) = \sum_{k=0}^{\infty}\frac{(b-c+1)_k B(b+k,1)}{k!\,B(b,c-b)} F_p^{(\alpha,\beta,\gamma)}(a,b+k;b+k+1;z),$$
$$\Phi_p^{(\alpha,\beta,\gamma)}(b;c;z) = \sum_{k=0}^{\infty}\frac{(b-c+1)_k B(b+k,1)}{k!\,B(b,c-b)} \Phi_p^{(\alpha,\beta,\gamma)}(b+k;b+k+1;z),$$
$$F_p^{(\alpha,\beta,\gamma)}(a,b;c;z) = \sum_{n=0}^{k}\binom{k}{n}\frac{B(b+n,c-b-n+k)}{B(b,c-b)} F_p^{(\alpha,\beta,\gamma)}(a,b+n;c+k;z),$$
$$\Phi_p^{(\alpha,\beta,\gamma)}(b;c;z) = \sum_{n=0}^{k}\binom{k}{n}\frac{B(b+n,c-b-n+k)}{B(b,c-b)} \Phi_p^{(\alpha,\beta,\gamma)}(b+n;c+k;z).$$

Using the relations [9] $B(b,c-b) = \frac{(c)_k}{(b)_k} B(b+k,c-b)$, $(a)_{n+k} = (a)_n(a+n)_k$, and $\frac{d^k}{dz^k} z^n = \frac{n!}{(n-k)!} z^{n-k}$, we get the following result

**Theorem 4.** *For $F_p^{(\alpha,\beta,\gamma)}(a,b;c;z)$ and $\Phi_p^{(\alpha,\beta,\gamma)}(b;c;z)$, the following differentiation formulas hold*

$$\frac{d^k}{dz^k}\left\{F_p^{(\alpha,\beta,\gamma)}(a,b;c;z)\right\} = \frac{(a)_k(b)_k}{(c)_k} F_p^{(\alpha,\beta,\gamma)}(a+k,b+k;c+k;z),$$

$$\frac{d^k}{dz^k}\left\{\Phi_p^{(\alpha,\beta,\gamma)}(b;c;z)\right\} = \frac{(b)_k}{(c)_k}\Phi_p^{(\alpha,\beta,\gamma)}(b+k;c+k;z).$$

## 3. Integral representations

In this section, some integral representations for $F_p^{(\alpha,\beta,\gamma)}(a,b;c;z)$ and $\Phi_p^{(\alpha,\beta,\gamma)}(b;c;z)$ are given in the form of the following theorems:

**Theorem 5.** *For $F_p^{(\alpha,\beta,\gamma)}(a,b;c;z)$, the following integral representations hold;*

$$F_p^{(\alpha,\beta,\gamma)}(a,b;c;z) = \frac{1}{B(b,c-b)}\int_0^1 u^{b-1}(1-u)^{c-b-1}(1-zu)^{-a}\,E_{\alpha,\beta}^{\gamma}\left(\frac{-p}{u(1-u)}\right)du, \tag{22}$$

$$F_p^{(\alpha,\beta,\gamma)}(a,b;c;z) = \frac{1}{B(b,c-b)}\int_0^\infty u^{b-1}(1+u)^{a-c}[1+u(1-z)]^{-a}E_{\alpha,\beta}^{\gamma}\left(-p\left(2+u+\frac{1}{u}\right)\right)du, \tag{23}$$

$$F_p^{(\alpha,\beta,\gamma)}(a,b;c;z) = \frac{2}{B(b,c-b)}\int_0^{\frac{\pi}{2}} \sin^{2b-1}u\cos^{2c-2b-1}u\left(1-z\sin^2 u\right)^{-a}E_{\alpha,\beta}^{\gamma}\left(-p\sec^2 u\csc^2 u\right)du, \tag{24}$$

$$F_p^{(\alpha,\beta,\gamma)}(a,b;c;z) = \frac{(m-l)^{1+a-c}}{B(b,c-b)}\int_l^m (u-l)^{b-1}(m-u)^{c-b-1}[m-l-z(u-l)]^{-a}E_{\alpha,\beta}^{\gamma}\left(\frac{-p(m-l)^2}{(u-l)(m-u)}\right)du, \tag{25}$$

$$F_p^{(\alpha,\beta,\gamma)}(a,b;c;z) = \frac{w^{1+a-c}}{B(b,c-b)}\int_0^w u^{b-1}(w-u)^{c-b-1}(w-zu)^{-a}E_{\alpha,\beta}^{\gamma}\left(\frac{-pw^2}{u(1-u)}\right)du, \tag{26}$$

$$F_p^{(\alpha,\beta,\gamma)}(a,b;c;z) = \frac{2^{1+a-c}}{B(b,c-b)}\int_{-1}^1 (1+u)^{b-1}(1-u)^{c-b-1}[2-z(1+u)]^{-a}E_{\alpha,\beta}^{\gamma}\left(\frac{-4p}{1-u^2}\right)du, \tag{27}$$

$$F_p^{(\alpha,\beta,\gamma)}(a,b;c;z) = \frac{1}{B(b,c-b)}\int_0^1\left[\frac{u^{b-1}(1+u(1-z))^{-a}+u^{c-b-1}(1+u-z)^{-a}}{(1+u)^{c-a}}\right]E_{\alpha,\beta}^{\gamma}\left(-p\left(2+u+\frac{1}{u}\right)\right)du, \tag{28}$$

$$F_p^{(\alpha,\beta,\gamma)}(a,b;c;z) = \frac{2^{1-c}}{B(b,c-b)}\int_{-\infty}^\infty \frac{\exp\left(u[2b-c]\right)\left(\cosh u-\frac{z}{2}e^t\right)^{-a}}{\cosh^{c-a}u}E_{\alpha,\beta}^{\gamma}\left(-4p\cosh^2 u\right)du. \tag{29}$$

**Proof.** Using

$$B_p^{(\alpha,\beta,\gamma)}(x,y) = \int_0^1 u^{x-1}(1-u)^{y-1}\,E_{\alpha,\beta}^{\gamma}\left(\frac{-p}{u(1-u)}\right)du, \tag{30}$$

$$B_p^{(\alpha,\beta,\gamma)}(x,y) = \int_0^\infty \frac{u^{x-1}}{(1+u)^{x+y}}\,E_{\alpha,\beta}^{\gamma}\left(-p\left(2+u+\frac{1}{u}\right)\right)du, \tag{31}$$

$$B_p^{(\alpha,\beta,\gamma)}(x,y) = 2\int_0^{\frac{\pi}{2}} \cos^{2x-1}u\sin^{2y-1}u\,E_{\alpha,\beta}^{\gamma}(-p\sec^2 u\csc^2 u)\,du, \tag{32}$$

$$B_p^{(\alpha,\beta,\gamma)}(x,y) = (m-l)^{1-x-y}\int_a^b (u-l)^{x-1}(m-u)^{y-1}\,E_{\alpha,\beta}^{\gamma}\left(\frac{-p(m-l)^2}{(u-l)(m-u)}\right)du, \tag{33}$$

$$B_p^{(\alpha,\beta,\gamma)}(x,y) = w^{1-x-y}\int_0^w u^{x-1}(w-u)^{y-1}E_{\alpha,\beta}^{\gamma}\left(\frac{-pw^2}{u(w-u)}\right)du, \tag{34}$$

$$B_p^{(\alpha,\beta,\gamma)}(x,y) = 2^{1-x-y}\int_{-1}^1 (1+u)^{x-1}(1-u)^{y-1}\,E_{\alpha,\beta}^{\gamma}\left(\frac{-4p}{1-u^2}\right)du, \tag{35}$$

$$B_p^{(\alpha,\beta,\gamma)}(x,y) = \int_0^1 \frac{u^{x-1}+u^{y-1}}{(1+u)^{x+y}}\,E_{\alpha,\beta}^{\gamma}\left(-p\left(2+u+\frac{1}{u}\right)\right)du, \tag{36}$$

$$B_p^{(\alpha,\beta,\gamma)}(x,y) = 2^{1-x-y}\int_{-\infty}^\infty \frac{e^{u(x-y)}}{(\cosh u)^{x+y}}\,E_{\alpha,\beta}^{\gamma}\left(-4p\cosh^2 u\right)du, \tag{37}$$

in (9) respectively, we get the desired results (22)-(29). □

Patting $u=\tanh^2 t$ in assertion (22) of Theorem 5, we get the following result:

**Corollary 1.** *For $F_p^{(\alpha,\beta,\gamma)}(a,b;c;z)$, the following integral representation holds;*

$$F_p^{(\alpha,\beta,\gamma)}(a,b;c;z) = \frac{2}{B(b,c-b)} \int_0^\infty \frac{(\sinh t)^{2b-1}(\cosh t)^{2a-2c+1}}{(\cosh^2 t - z\sinh^2 t)^a} E_{\alpha,\beta}^{\gamma}\left(-p\cosh^2 t \coth^2 t\right) dt.$$

**Theorem 6.** *For $\Phi_p^{(\alpha,\beta,\gamma)}(b;c;z)$, the following integral representations hold;*

$$\Phi_p^{(\alpha,\beta,\gamma)}(b;c;z) = \frac{1}{B(b,c-b)} \int_0^1 u^{b-1}(1-u)^{c-b-1} e^{zu} E_{\alpha,\beta}^{\gamma}\left(\frac{-p}{u(1-u)}\right) du, \tag{38}$$

$$\Phi_p^{(\alpha,\beta,\gamma)}(b;c;z) = \frac{1}{B(b,c-b)} \int_0^\infty \frac{u^{b-1}}{(1+u)^c} \exp\left(\frac{zu}{1+u}\right) E_{\alpha,\beta}^{\gamma}\left(-p\left(2+u+\frac{1}{u}\right)\right) du, \tag{39}$$

$$\Phi_p^{(\alpha,\beta,\gamma)}(b;c;z) = \frac{2}{B(b,c-b)} \int_0^{\frac{\pi}{2}} \sin^{2b-1} u \cos^{2c-2b-1} u\, e^{z\sin^2 u} E_{\alpha,\beta}^{\gamma}\left(-p\sec^2 u \csc^2 u\right) du \tag{40}$$

$$\Phi_p^{(\alpha,\beta,\gamma)}(b;c;z) = \frac{e^{l-m}(m-l)^{1-c}}{B(b,c-b)} \int_l^m (u-l)^{b-1}(m-u)^{c-b-1} e^{z(u-l)} E_{\alpha,\beta}^{\gamma}\left(\frac{-p(m-l)^2}{(u-l)(m-u)}\right) du, \tag{41}$$

$$\Phi_p^{(\alpha,\beta,\gamma)}(b;c;z) = \frac{w^{1-c}}{B(b,c-b)} \int_0^w u^{b-1}(w-u)^{c-b-1} e^{\frac{zu}{w}} E_{\alpha,\beta}^{\gamma}\left(\frac{-pw^2}{u(1-u)}\right) du, \tag{42}$$

$$\Phi_p^{(\alpha,\beta,\gamma)}(b;c;z) = \frac{2^{1-c}\, e^2}{B(b,c-b)} \int_{-1}^1 (u+1)^{b-1}(1-u)^{c-b-1} e^{z(u+1)} E_{\alpha,\beta}^{\gamma}\left(\frac{-4p}{1-u^2}\right) du, \tag{43}$$

$$\Phi_p^{(\alpha,\beta,\gamma)}(b;c;z) = \frac{1}{B(b,c-b)} \int_0^1 \left[\frac{u^{b-1}\exp(\frac{zu}{1+u}) + u^{c-b-1}\exp(\frac{z}{1+u})}{(1+u)^c}\right] E_{\alpha,\beta}^{\gamma}\left(-p\left(2+u+\frac{1}{u}\right)\right) du, \tag{44}$$

$$\Phi_p^{(\alpha,\beta,\gamma)}(b;c;z) = \frac{2^{1-c}\, e^2}{B(b,c-b)} \int_{-\infty}^\infty \frac{\exp\left[(\frac{ze^u}{\cosh u}) + u(2b-c)\right]}{\cosh^c u} E_{\alpha,\beta}^{\gamma}\left(-4p\cosh^2 u\right) du. \tag{45}$$

**Proof.** Using (30)-(37) in (10), respectively, we get the desired results (38)-(45). □

Patting $u = \tanh^2 t$ in assertion (38) of Theorem 6, we get the following result:

**Corollary 2.** *For $Phi_p^{(\alpha,\beta,\gamma)}(b;c;z)$, the following integral representation holds;*

$$\Phi_p^{(\alpha,\beta,\gamma)}(b;c;z) = \frac{2e^z}{B(b,c-b)} \int_0^\infty (\sinh u)^{2b-1}(\cosh u)^{1-2c} \tanh^2 u E_{\alpha,\beta}^{\gamma}\left(-p\cosh^2 u \coth^2 u\right) du.$$

**Theorem 7.** *For $F_p^{(\alpha,\beta,\gamma)}(a,b;c;z)$ and $\Phi_p^{(\alpha,\beta,\gamma)}(b;c;z)$, we have*

$$F_p^{(\alpha,\beta,\gamma)}(a,b;c;z) = \frac{1}{\alpha\,\Gamma(\beta-\alpha)} \int_0^1 F_{pu}^{(\alpha,\alpha,\gamma)}(a,b;c;z)\left(1-u^{\frac{1}{\alpha}}\right)^{\beta-\alpha-1} du, \tag{46}$$

*and*

$$\Phi_p^{(\alpha,\beta,\gamma)}(b;c;z) = \frac{1}{\alpha\,\Gamma(\beta-\alpha)} \int_0^1 \Phi_{pu}^{(\alpha,\alpha,\gamma)}(b;c;z)\left(1-u^{\frac{1}{\alpha}}\right)^{\beta-\alpha-1} du. \tag{47}$$

**Proof.** Since

$$B_p^{(\alpha,\beta,\gamma)}(x,y) = \frac{1}{\alpha\,\Gamma(\beta-\alpha)} \int_0^1 B_{pu}^{(\alpha,\alpha,\gamma)}(x,y)\left(1-u^{\frac{1}{\alpha}}\right)^{\beta-\alpha-1} du, \tag{48}$$

$$F_p^{(\alpha,\beta,\gamma)}(a,b;c;z) = \sum_{n=0}^\infty (a)_n \int_0^1 \frac{B_{pu}^{(\alpha,\alpha,\gamma)}(b+n,c-b)\left(1-u^{\frac{1}{\alpha}}\right)^{\beta-\alpha-1} du}{\alpha\,\Gamma(\beta-\alpha)B(b,c-b)} \frac{z^n}{n!}.$$

Rearranging the integration and summation, we get

$$F_p^{(\alpha,\beta,\gamma)}(a,b;c;z) = \frac{1}{\alpha\,\Gamma(\beta-\alpha)} \int_0^1 \left[ \sum_{n=0}^{\infty} (a)_n \frac{B_{pu}^{(\alpha,\alpha,\gamma)}(b+n,c-b)}{B(b,c-b)} \frac{z^n}{n!} \right] \left(1-u^{\frac{1}{\alpha}}\right)^{\beta-\alpha-1} du,$$

which on using (9), gives the desired result (46).

Similarly, using (48) in (10) and following the same procedure leading to result (46), we obtain (47) and thus the proof of Theorem 7 is completed. □

Putting $\mu = u^{\frac{1}{\alpha}}$ in assertions (46) and (47) of Theorem 7, we get the following results:

**Corollary 3.** *For $F_p^{(\alpha,\beta,\gamma)}(a,b;c;z)$ and $\Phi_p^{(\alpha,\beta,\gamma)}(b;c;z)$, we have*

$$F_p^{(\alpha,\beta,\gamma)}(a,b;c;z) = \frac{1}{\Gamma(\beta-\alpha)} \int_0^1 F_{p\mu^\alpha}^{(\alpha,\alpha,\gamma)}(a,b;c;z)\, \mu^{\alpha-1}(1-\mu)^{\beta-\alpha-1}\, d\mu,$$

*and*

$$\Phi_p^{(\alpha,\beta,\gamma)}(b;c;z) = \frac{1}{\Gamma(\beta-\alpha)} \int_0^1 \Phi_{p\mu^\alpha}^{(\alpha,\alpha,\gamma)}(b;c;z)\, \mu^{\alpha-1}(1-\mu)^{\beta-\alpha-1}\, d\mu.$$

Now, using the relations [13]

$$B_p^{(\alpha,\beta,\gamma)}(x,y) = \frac{1}{\Gamma(\alpha)} \int_0^1 B_{p(1-u)^\alpha}^{(\alpha,\beta-\alpha,\gamma)}(x,y)\, u^{\alpha-1}(1-u)^{\beta-\alpha-1}\, du,$$

$$(\alpha,\beta,\gamma \in \mathbb{C};\ \ Re(\beta) > Re(\alpha) > 0,\ \ Re(\gamma) > 0),$$

$$B_p^{(\alpha,\beta,\gamma)}(x,y) = \frac{1}{B(\gamma,l-\gamma)} \int_0^1 B_{pu}^{(\alpha,\beta,l)}(x,y)\, u^{\gamma-1}(1-u)^{l-\gamma-1}\, du,$$

$$(\alpha,\beta,\gamma,l \in \mathbb{C};\ \ Re(l) > Re(\gamma) > 0,\ \ Re(\alpha) > 0,\ \ Re(\beta) > 0),$$

$$B_p^{(\alpha,\beta+l,\gamma)}(x,y) = \frac{1}{\Gamma(l)} \int_0^1 B_{pu^\alpha}^{(\alpha,\beta,\gamma)}(x,y)\, u^{\beta-1}(1-u)^{l-1}\, du,$$

$$(\alpha,\beta,\gamma,l \in \mathbb{C};\ \ Re(\alpha) > 0,\ \ Re(\beta) > 0,\ \ Re(\gamma) > 0, Re(l) > 0)$$

and following the same procedure leading to the results in the above theorem, we get the following results:

**Theorem 8.** *For $F_p^{(\alpha,\beta,\gamma)}(a,b;c;z)$ and $\Phi_p^{(\alpha,\beta,\gamma)}(b;c;z)$, we have*

$$F_p^{(\alpha,\beta,\gamma)}(a,b;c;z) = \frac{1}{\Gamma(\alpha)} \int_0^1 F_{p(1-u)^\alpha}^{(\alpha,\beta-\alpha,\gamma)}(a,b;c;z)\, u^{\alpha-1}(1-u)^{\beta-\alpha-1}\, du,$$

$$\Phi_p^{(\alpha,\beta,\gamma)}(b;c;z) = \frac{1}{\Gamma(\alpha)} \int_0^1 \Phi_{p(1-u)^\alpha}^{(\alpha,\beta-\alpha,\gamma)}(b;c;z)\, u^{\alpha-1}(1-u)^{\beta-\alpha-1}\, du,$$

$$(Re(\beta) > Re(\alpha) > 0;\ \ Re(\gamma) > 0;\ \ Re(c) > Re(b) > 0).$$

$$F_p^{(\alpha,\beta,\gamma)}(a,b;c;z) = \frac{1}{B(\gamma,l-\gamma)} \int_0^1 F_{pu}^{(\alpha,\beta,l)}(a,b;c;z)\, u^{\gamma-1}(1-u)^{l-\gamma-1}\, du,$$

$$\Phi_p^{(\alpha,\beta,\gamma)}(b;c;z) = \frac{1}{B(\gamma,l-\gamma)} \int_0^1 \Phi_{pu}^{(\alpha,\beta,l)}(b;c;z)\, u^{\gamma-1}(1-u)^{l-\gamma-1}\, du,$$

$$(Re(l) > Re(\gamma) > 0;\ \ Re(\alpha) > 0;\ \ Re(\beta) > 0;\ \ Re(c) > Re(b) > 0).$$

$$F_p^{(\alpha,\beta+l,\gamma)}(a,b;c;z) = \frac{1}{\Gamma(l)}\int_0^1 F_{pu^\alpha}^{(\alpha,\beta,\gamma)}(a,b;c;z)\,u^{\beta-1}(1-u)^{l-1}\,du,$$
$$\Phi_p^{(\alpha,\beta+l,\gamma)}(b;c;z) = \frac{1}{\Gamma(l)}\int_0^1 \Phi_{pu^\alpha}^{(\alpha,\beta,\gamma)}(b;c;z)\,u^{\beta-1}(1-u)^{l-1}\,du,$$
$$(Re(\alpha)>0;\ Re(\beta)>0;\ Re(\gamma)>0;\ Re(l)>0;\ Re(c)>Re(b)>0).$$

**Theorem 9.** *For $F_p^{(\alpha,\beta,\gamma)}(a,b;c;z)$ and $\Phi_p^{(\alpha,\beta,\gamma)}(b;c;z)$, we have*

$$F_p^{(k,\beta,\gamma)}(a,b;c;z) = \frac{1}{\Gamma(\beta)B(b,c-b)}\int_0^1 u^{b-1}(1-u)^{c-b-1}(1-zu)^{-a}{}_1F_k\left[\begin{matrix}\gamma & ; \\ \frac{\beta}{k},\frac{\beta+1}{k},\ldots,\frac{\beta+k-1}{k}; \end{matrix}\ \frac{-p}{k^k u(1-u)}\right]du,$$
$$\Phi_p^{(k,\beta,\gamma)}(b;c;z) = \frac{1}{\Gamma(\beta)B(b,c-b)}\int_0^1 u^{b-1}(1-u)^{c-b-1}e^{zu}{}_1F_k\left[\begin{matrix}\gamma & ; \\ \frac{\beta}{k},\frac{\beta+1}{k},\ldots,\frac{\beta+k-1}{k}; \end{matrix}\ \frac{-p}{k^k u(1-u)}\right]du.$$

**Proof.** Using the known relation [13]

$$B_p^{(k,\beta,\gamma)}(x,y) = \frac{1}{\Gamma(\beta)}\int_0^1 u^{x-1}(1-u)^{y-1}{}_1F_k\left[\begin{matrix}\gamma & ; \\ \frac{\beta}{k},\frac{\beta+1}{k},\ldots,\frac{\beta+k-1}{k}; \end{matrix}\ \frac{-p}{k^k u(1-u)}\right]du, \tag{49}$$

in (9) (for $\alpha = k$), we get

$$F_p^{(k,\beta,\gamma)}(a,b;c;z) = \frac{1}{\Gamma(\beta)B(b,c-b)}\sum_{n=0}^{\infty}(a)_n\int_0^1 u^{b+n-1}(1-u)^{c-b-1}$$
$$\times\,{}_1F_k\left[\begin{matrix}\gamma & ; \\ \frac{\beta}{k},\frac{\beta+1}{k},\ldots,\frac{\beta+k-1}{k}; \end{matrix}\ \frac{-p}{k^k u(1-u)}\right]\frac{z^n}{n!}\,du.$$

Rearranging the integration and summation, we get

$$F_p^{(k,\beta,\gamma)}(a,b;c;z) = \frac{1}{\Gamma(\beta)B(b,c-b)}\int_0^1 u^{b-1}(1-u)^{c-b-1}$$
$$\times\,{}_1F_k\left[\begin{matrix}\gamma & ; \\ \frac{\beta}{k},\frac{\beta+1}{k},\ldots,\frac{\beta+k-1}{k}; \end{matrix}\ \frac{-p}{k^k u(1-u)}\right]\sum_{n=0}^{\infty}(a)_n\frac{(zu)^n}{n!}\,du,$$

which on using [9]

$$(1-t)^{-\alpha} = \sum_{n=0}^{\infty}\frac{(\alpha)_n}{n!}t^n, \qquad (|t|<1),$$

in the right hand side yields the desired result (49).

Similarly, using (49) in (10) and following the same procedure leading to result (49), we obtain result (49) and thus the proof of Theorem 9 is completed. □

## 4. Some transformation formulas

First, the Mellin transform representation for the new extended Gauss hypergeometric and confluent hypergeometric functions are obtained in the form of the following theorem:

**Theorem 10.** *For $F_p^{(\alpha,\beta,\gamma)}(a,b;c;z)$ and $\Phi_p^{(\alpha,\beta,\gamma)}(b;c;z)$, we have*

$$\mathcal{M}\left\{F_p^{(\alpha,\beta,\gamma)}(a,b;c;z);s\right\} = \frac{\Gamma_0^{(\alpha,\beta,\gamma)}(s)B(b+s,c+s-b)}{B(b,c-b)}{}_2F_1(a,b+s;c+2s;z), \tag{50}$$

$$\mathcal{M}\left\{\Phi_p^{(\alpha,\beta,\gamma)}(b;c;z);s\right\} = \frac{\Gamma_0^{(\alpha,\beta,\gamma)}(s)B(b+s,c+s-b)}{B(b,c-b)}{}_1F_1(b+s;c+2s;z). \tag{51}$$

**Proof.** Using the definition of Mellin transform, we get

$$\mathcal{M}\left\{F_p^{(\alpha,\beta,\gamma)}(a,b;c;z);s\right\} = \int_0^\infty p^{s-1}F_p^{(\alpha,\beta,\gamma)}(a,b;c;z)\,dp.$$

Now using relation (22) in the right hand side, we get

$$\mathcal{M}\left\{F_p^{(\alpha,\beta,\gamma)}(a,b;c;z);s\right\} = \frac{1}{B(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}\left[\int_0^\infty p^{s-1}E_{\alpha,\beta}^{\gamma}\left(\frac{-p}{t(1-t)}\right)dp\right]dt.$$

Now using the one-to-one transformation (except possibly at the boundaries and maps the region onto itself) $u=\frac{p}{t(1-t)}$, $w=t$ in the above equation and considering that the Jacobian of the transformation is $J=w(1-w)$, we get

$$\begin{aligned}&\mathcal{M}\left\{F_p^{(\alpha,\beta,\gamma)}(a,b;c;z);s\right\}\\ &= \frac{B(b+s,c+s-b)}{B(b,c-b)}\frac{1}{B(b+s,c+s-b)}\int_0^1 w^{b+s-1}(1-w)^{c+s-b-1}(1-zw)^{-a}\,dw \times \int_0^\infty u^{s-1}E_{\alpha,\beta}^{\gamma}(-u)\,du,\end{aligned}$$

which on using relations (3) and (7) (for $p=0$) in the right hand side yields the desired result (50).

Similarly, following the same procedure leading to result, we obtain result (51) and thus the proof of Theorem 10 is completed. □

**Theorem 11.** *For $F_p^{(\alpha,\beta,\gamma)}(a,b;c;z)$ and $\Phi_p^{(\alpha,\beta,\gamma)}(b;c;z)$, we have*

$$F_p^{(\alpha,\beta,\gamma)}(a,b;c;z) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{\Gamma_0^{(\alpha,\beta,\gamma)}(s)B(b+s,c+s-b)}{B(b,c-b)}{}_2F_1(a,b+s;c+2s;z)p^{-s}\,ds,$$

$$\Phi_p^{(\alpha,\beta,\gamma)}(b;c;z) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{\Gamma_0^{(\alpha,\beta,\gamma)}(s)B(b+s,c+s-b)}{B(b,c-b)}{}_1F_1(b+s;c+2s;z)p^{-s}\,ds.$$

**Proof.** Taking Mellin inversion of Theorem 10, we get the desired results. □

Next, we prove some other transformation formulas in the form of the following theorems:

**Theorem 12.** *For $F_p^{(\alpha,\beta,\gamma)}(a,b;c;z)$, we have*

$$F_p^{(\alpha,\beta,\gamma)}(a,b;c;z) = (1-z)^{-a}F_p^{(\alpha,\beta,\gamma)}\left(a,c-b;c;\frac{z}{z-1}\right), \tag{52}$$

$$F_p^{(\alpha,\beta,\gamma)}(a,b;c;z) = (1-z)^{-b}F_p^{(\alpha,\beta,\gamma)}\left(c-a,b;c;\frac{z}{z-1}\right), \tag{53}$$

$$F_p^{(\alpha,\beta,\gamma)}(a,b;c;z) = (1-z)^{c-a-b}F_p^{(\alpha,\beta,\gamma)}(c-a,c-b;c;z), \qquad (|\arg(1-z)|<\pi). \tag{54}$$

**Proof.** By writing

$$[1-z(1-t)]^{-a} = (1-z)^{-a}\left(1+\frac{z}{1-z}t\right)^{-a},$$

and replacing $u \to 1-t$ in (22), we get

$$F_p^{(\alpha,\beta,\gamma)}(a,b;c;z) = \frac{(1-z)^{-a}}{B(b,c-b)} \int_0^1 (1-t)^{b-1} t^{c-b-1} \left(1 - \frac{z}{z-1} t\right)^{-a} \times E_{\alpha,\beta}^{\gamma}\left(\frac{-p}{t(1-t)}\right) dt,$$

which on using (22), gives the desired result (52).

Similarly, using the same steps above and following the same procedure leading to result (52), we get the desired result (53).

Now from (52) and (53), we obtain

$$(1-z)^{-a} F_p^{(\alpha,\beta,\gamma)}\left(a, c-b; c; \frac{z}{z-1}\right) = (1-z)^{-b} F_p^{(\alpha,\beta,\gamma)}\left(c-a, b; c; \frac{z}{z-1}\right),$$

which on putting $a = c-a$ and $z = \frac{z}{z-1}$, we get the desired result (54) and thus the proof of Theorem 12 is completed. □

**Theorem 13.** *For $\Phi_p^{(\alpha,\beta,\gamma)}(b;c;z)$, we have*

$$\Phi_p^{(\alpha,\beta,\gamma)}(b;c;z) = \exp(z) \Phi_p^{(\alpha,\beta,\gamma)}(c-b;c;-z).$$

**Proof.** Putting $u = 1-t$ in (38), we obtain

$$\Phi_p^{(\alpha,\beta,\gamma)}(b;c;z) = \frac{1}{B(b,c-b)} \int_0^1 (1-t)^{b-1} t^{c-b-1} e^{z(1-t)} E_{\alpha,\beta}^{\gamma}\left(\frac{-p}{t(1-t)}\right) dt,$$

which on using some simplification and using (38), yields the desired result. □

**Theorem 14.** *For $\Phi_p^{(\alpha,\beta,\gamma)}(b;c;z)$, we have the following relation*

$$\int_0^\infty z^{a-1} \Phi_p^{(\alpha,\beta,\gamma)}(b;c;-z)\, dz = \Gamma(a) F_p^{(\alpha,\beta,\gamma)}(a, c-b; c; 1), \tag{55}$$

$$(Re(\alpha) > 0;\; Re(\beta) > 0;\; Re(\gamma) > 0;\; Re(p) > 0;\; Re(a) > 0).$$

**Proof.** From Theorem 13 for $z = -z$, we obtain

$$\Phi_p^{(\alpha,\beta,\gamma)}(b;c;-z) = \exp(-z) \Phi_p^{(\alpha,\beta,\gamma)}(c-b;c;z). \tag{56}$$

Multiplying both sides by $z^{a-1}$ and integrating the resultant equation with respect to $z$ from $z = 0$ to $z = \infty$, we get

$$\begin{aligned} \int_0^\infty z^{a-1} \Phi_p^{(\alpha,\beta,\gamma)}(b;c;-z)\, dz &= \int_0^\infty z^{a-1} e^{-z} \Phi_p^{(\alpha,\beta,\gamma)}(c-b;c;z)\, dz, \\ &= \int_0^\infty z^{a-1} e^{-z} \sum_{n=0}^{\infty} \frac{B_p^{(\alpha,\beta,\gamma)}(c-b+n,b)}{B(b,c-b)} \frac{z^n}{n!}\, dz. \end{aligned}$$

Interchanging the order of integration and summation, we get

$$\int_0^\infty z^{a-1} \Phi_p^{(\alpha,\beta,\gamma)}(b;c;-z)\, dz = \sum_{n=0}^{\infty} \frac{B_p^{(\alpha,\beta,\gamma)}(c-b+n,b)}{B(b,c-b)n!} \int_0^\infty z^{n+a-1} e^{-z}\, dz,$$

which on using definition of Euler Gamma function [9] and then using (9), yields the desired result. □

## 5. Concluding remarks

In this paper, the authors established new extension forms of the hypergeometric functions with the help of the new definition of extended Beta function given in [13]. Also, various properties of this extended functions are obtained. The authors conclude that if we let $\alpha = \beta = \gamma = 1$ throughout in the paper and use the relations (13) and (17), then some known and new results due to the work of Chaudhry *et al.*, [2] will be obtained. Also, if we let $\alpha = 1$ throughout in the paper and use the relations (11) and (15) then some known results due to work of Özergin *et al.*, [6] will be obtained. Further if we let $\beta = \gamma = 1$ throughout in the paper and use the relations (12) and (16), then some known results due to work of Shadab *et al.*, [11] will be obtained.

**Author Contributions:** All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

**Conflicts of Interest:** "The authors declare no conflict of interest."

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