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Moments of Truncated Skew-t Distribution and Graph Theory Applied to the Shortest Path Problem

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Abstract

In the shortest path problem most approaches has been proposed over the last twenty years are focused to deterministic approaches. Stochastic approaches that include theory of truncated asymmetric probability distributions have not been tackled in the literature of optimal paths. Since, in practice, the paths are distances that must be traveled in finite times which are not always fixed, the stochasticity of the time has to be considered into the problem. In this paper, we consider using the moments of the truncated skew-t distribution to the problem of finding the shortest path between two locations with minimum distance by the transition times. The skew-tand truncated skew-t distributions are described explicitly to show the moments and their existence by the convergence of the hypergeometric series. An application to optimal paths using the moments of the truncated skew-t distribution and the graph theory illustrates the shortest path by the minimum average transition time.

Keywords: Skew-t; moments; truncated distributions; hypergeometric series; graph theory.

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1 Introduction

In the literature, truncated moments have been derived for several truncated distributions, particularly for the doubly truncated distributions, for instance in [1] is considered the truncated moments of a doubly truncated binomial distribution. Truncated moments of a doubly truncated normal were derived in [2]. Sugiura and Gomi [3] studied the skewness and kurtosis of a doubly truncated normal and a right truncated Weibull distribution. Recently, in [4] was derived several expressions for truncated moments useful for numerical computation. These include truncated moments of normal, lognormal, Pearson type III, log Pearson type III, and extreme value (Weibull and Gumbel) distributions.

In the last decade the interest has focused on finding more flexible ways to represent the data, avoiding unrealistic assumptions [5]. A flexible class of location-scale models are defined by the skew-elliptical family of distributions; see [6], [7, 8], [9], [10], [11]. This class allows modeling skewness in the distribution of the data, however the models must be based on reals characteristics of the data, for instance proposing skew models. In [12] provides an overview of the literature concerned with the family of skew-normal distributions and other continuous random variables families linked to class skew-normal. The goodness-of-fit tests for the skew-normal distribution were proposed in [13], [14], where a characterization of the skew-normal as well as relevant results on quadratic and linear forms are given; properties of the skew-normal, skew-uniform, skew-t, skew-Cauchy, skew-Laplace, and the skew-logistic distributions were explored in [15]. Later, in [16], and [17], were studied some asymmetric and symmetric multivariate distributions. On a specific problem of climatic data, in [18] was shown the need to study the truncation of the skew-normal, and also the need to study the simultaneity of the asymmetry and truncation, indicating the existence of natural physical limits, that is, the philosophy of modeling started to change.

In this paper is proposed to use the moments of the truncated skew-t distribution which are applied to the common problem of optimal paths. Most calculations of the moments are based in [19] and the generalizations in [20], specifying the conditions of convergence of the hypergeometric series. Finally based on [6], [8], [16], [21], is presented the probability density function and cumulative distribution function of skew-t distribution, and the moments and the truncated density are characterized. The main objective of this paper is twofold: to study the convergence of the hypergeometric series to guarantee the existence of the moments of a truncated skew-t distribution; and to applied the moments of such distribution to the path problem in order to define finite traveling times to find the shortest path between two locations using the graph theory.

The outline of this paper is as follows. In Section 2, formally describe the skew-t distribution. Section 3, introduces the truncated skew-t distribution and the conditions to define the moments of the distribution. Section 4, presents the study of convergence of the hypergeometric series to ensure the existence of the moments of the distribution. An application to optimal path is considered in Section 5. Finally in Section 6 some concluding remarks are discussed.

2 The Skew-t Distribution

Lemma 2.1. *If f_0 is a one-dimensional probability density function symmetric about 0, and G is a one–dimensional distribution function such that $G^{'}$ exists and is a density symmetric about 0, then for $-\infty < z < \infty$, $f(z) = 2f_0(z)G(w(z))$ is a density function for any odd function $w(.)$ ([12]).*

Proof. If $Y \sim f_0$ and $X \sim G^{'}$ are independent random variables, then

$$\frac{1}{2} = P(X - w(Y) \leq 0) = E_Y(X - w(Y) \leq 0 \mid Y) = \int G\left\{w(z)\right\} f_0(z)dz,$$

on noticing that $w(Y)$ and $X - w(Y)$ also have symmetric distribution about 0 ([12]). □

Using lemma 2.1, where $f_0 = t_1(x,v)$ and $G = T_1\left(\alpha x\left(\frac{v+1}{v+x^2}\right)^{\frac{1}{2}}; v+1\right)$, the density function of a skew-t can be represented by:

$$f_X(x;\alpha,v) = 2t_1(x;v)T_1\left(\alpha x\left(\frac{v+1}{v+x^2}\right)^{\frac{1}{2}}; v+1\right). \quad (2.1)$$

So X stands for a skew-t (ST) distribution with mean 0, skewness parameter α, degrees of freedom ν and density denoted by $ST(\alpha,\nu)$. Now, if $Y = \xi + \varpi X$, where $\xi \in IR$, $\varpi \in IR^+$, we obtain $Y \sim ST\left(\xi,\varpi^2,\alpha,\nu\right)$, and the probability density function (pdf) of Y, given by

$$f_Y(y,\xi,\varpi^2,\alpha,v) = 2t_1(y,\xi,\varpi^2,v)T_1\left(\alpha z\left(\frac{v+1}{v+z^2}\right)^{\frac{1}{2}}; v+1\right), \quad (2.2)$$

with $z = (y-\xi)/\varpi$.

Let $w(z,v) = z\left(\frac{v+1}{v+z^2}\right)^{\frac{1}{2}}$, we have that the pdf [22] is

$$f_Y(y,\xi,\varpi^2,\alpha,v) = 2t_1(y,\xi,\varpi^2,v)T_1\left(\alpha w(z,v); v+1\right), \quad (2.3)$$

explicitly,

$$\begin{aligned} f_X(x;\alpha,v) &= \frac{\Gamma(\frac{v+1}{2})}{\Gamma(\frac{v}{2})\sqrt{v\pi}}\left(1+\frac{x^2}{v}\right)^{-(\frac{v+1}{2})} \\ &\quad \int_{-\infty}^{x\alpha\sqrt{\frac{v+1}{v+x^2}}} \frac{\Gamma(\frac{v+2}{2})}{\Gamma(\frac{v+1}{2})\sqrt{(v+1)\pi}}\left(1+\frac{x^2}{v+1}\right)^{-(\frac{v+2}{2})} dx \\ &= \frac{(v+1)^{(\frac{v+1}{2})}v^{(\frac{v+2}{2})}}{2\pi}\left(v+x^2\right)^{-(\frac{v+1}{2})} \\ &\quad \int_{-\infty}^{x\alpha\sqrt{\frac{v+1}{v+x^2}}} \left[(v+1)+x^2\right]^{-(\frac{v+2}{2})} dx, \end{aligned}$$

with $v = 2, \ldots$ and $-\infty < x < \infty$.

Let

$$\int_{-\infty}^{x\alpha\sqrt{\frac{v+1}{v+x^2}}} \left[(v+1)+x^2\right]^{-(\frac{v+2}{2})} dx = 1 - W\left(x\alpha\sqrt{\frac{v+1}{v+x^2}}; v+1; \alpha\right),$$

thus,

$$f_X(x;\alpha,v) = \frac{(v+1)^{(\frac{v+1}{2})}v^{(\frac{v+2}{2})}}{2\pi}\left(v+x^2\right)^{-(\frac{v+1}{2})}\left[1 - W\left(x\alpha\sqrt{\frac{v+1}{v+x^2}}; v+1; \alpha\right)\right], \quad (2.4)$$

with $v = 2, \ldots$ and $-\infty < x < \infty$, where

$$\begin{aligned} W\left(x\alpha\sqrt{\frac{v+1}{v+x^2}}; v+1; \alpha\right) &= \left(x\alpha\sqrt{\frac{v+1}{v+x^2}}\right)^{-v} B(1,v) \\ &\quad {}_3F_2\left(\frac{v+1}{2}; \frac{v}{2}; \frac{v+1}{2}; \frac{v+1}{2}; \frac{v+2}{2}; \frac{-v+1}{\left(x\alpha\sqrt{\frac{v+1}{v+x^2}}\right)^2}\right), \end{aligned}$$

with ${}_3F_2$ being the generalized hypergeometric series [23]. Thus the skew-t distribution is defined by

$$f_X(x;\alpha,v) \quad = \frac{(v+1)^{\left(\frac{v+1}{2}\right)}v^{\left(\frac{v+2}{2}\right)}}{2\pi}\left(v+x^2\right)^{-\left(\frac{v+1}{2}\right)}\left[1-W\left(x\alpha\sqrt{\frac{v+1}{v+x^2}};v+1;\alpha\right)\right], \tag{2.5}$$

with $v = 2, ...$ and $-\infty < x < \infty$.

Fig. 1 shows the density of the skew-t distribution for increasing values of the skewness parameter α and a fixed degrees of freedom ν. The t distribution ($\alpha = 0$) is shown as a green line for comparison.

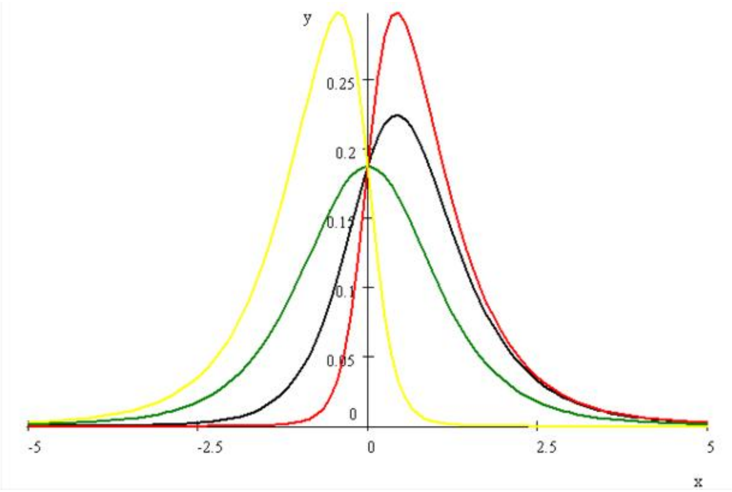

Fig. 1. Density curves of the skew-t distribution for several α values. Source: own research.

3 The Truncated Skew-t Distribution

Let X be a random variable with probability density function (pdf) $f(x)$ and the cumulative distribution function (cdf) $F(x)$, which both functions have infinity support. We have interest in compute the pdf of X in a restricted support $y = (a, b]$, that is, we want to compute X given $a < X \leq b$. This pdf is given by

$$f(x|a < X \leq b) = \frac{g(x)}{F(a) - F(b)} = Tr(x), \tag{3.1}$$

where $g(x) = f(x)$ for all $a < X \leq b$ and $g(x) = 0$ elsewhere. It should be noted that $Tr(x)$ has the same support of $g(x)$, and surely $f(x|a < X \leq b)$ is a distribution, $\int_a^b f(x|a < X \leq b)\,dx = \frac{1}{F(b)-F(a)}\int_a^b g(x)\,dx = 1$.

In our particular case, we consider the case that the random variable X is distributed as $ST(\alpha, \nu)$, with pdf $t_1(x, v)$ and cdf $T_1\left(\alpha x\left(\frac{v+1}{v+x^2}\right)^{\frac{1}{2}}; v+1\right)$, which both functions have infinity support.

Now to compute the truncated skew-t distribution, we need the Equation (3.1), where $g(x) = t_1(x, v)$ for all $a < X \leq b$ and $g(x) = 0$ elsewhere, and $Tr(x)$ has the same support $g(x)$.

Therefore,

$$\begin{aligned} f(x|a < X \leq b) &= \frac{g(x)}{F(a)-F(b)} \\ &= \frac{(v+1)^{(\frac{v+1}{2})}v^{(\frac{v+2}{2})}}{2\pi(B-A)}(v+x^2)^{-(\frac{v+1}{2})} \\ &\quad [1-W(r;v+1;\alpha)]\, I_{(a,b]}(x), \end{aligned} \tag{3.2}$$

where $r = x\alpha\sqrt{\frac{v+1}{v+x^2}}$, $B = \int_{-\infty}^{b} \frac{(v+1)^{(\frac{v+1}{2})}v^{(\frac{v+2}{2})}}{2\pi}(v+x^2)^{-(\frac{v+1}{2})}[1-W(r;v+1;\alpha)]\,dx$ and $A = \int_{-\infty}^{a} \frac{(v+1)^{(\frac{v+1}{2})}v^{(\frac{v+2}{2})}}{2\pi}(v+x^2)^{-(\frac{v+1}{2})}[1-W(r;v+1;\alpha)]\,dx$, are constants.

Thus the truncated skew-t distribution (tST) density is given by

$$\begin{aligned} f(x;\alpha,v) &= \frac{(v+1)^{(\frac{v+1}{2})}v^{(\frac{v+2}{2})}}{2\pi(B-A)}(v+x^2)^{-(\frac{v+1}{2})}[1-W(r;v+1;\alpha)]\, I_{(a,b]}(x) \\ &= \frac{k\left(v+x^2\right)^{-\left(\frac{v+1}{2}\right)} \int\limits_{-\infty}^{r} \left(v+1+t^2\right)^{-\left(\frac{v+2}{2}\right)} dt}{B-A} I_{(a,b]}(x), \end{aligned} \tag{3.3}$$

where $k = \frac{\Gamma\left(\frac{v+2}{2}\right)(v+1)^{\left(\frac{v+2}{2}\right)}}{\Gamma\left(\frac{v}{2}\right)\sqrt{(2v+1)\pi}}$.

Theorem 3.1. *Let $X \sim ST(\alpha,\nu)$, the probability density function of a truncated skew-t distribution exists if* $v < \frac{-\left[x^2(\alpha^2-1)+1\right]+\sqrt{\left[x^2(\alpha^2-1)+1\right]^2+4x^2(\alpha^2+1)}}{2}$.

Proof. The proof is straightforward based on the D'Alembert criterion for convergence, under the hypergeometric series ${}_3F_2\left(\frac{v+1}{2};\frac{v}{2};\frac{v+1}{2};\frac{v+1}{2};\frac{v+2}{2};\frac{-v+1}{r^2}\right)$, where $r = x\alpha\sqrt{\frac{v+1}{v+x^2}}$. □

A graphical representation of Theorem 3.1 is shown in Fig. 2, where the x axis represents the parameter of truncation and the y axis represents the skewness parameter. We can see from this figure that for small values of the truncation and skewness parameters the degrees of freedom is very limited, but for large values of the truncation parameter and small values of the skewness parameter the degrees of freedom is almost unrestricted.

3.1 The Moments

In this subsection we present the calculations of the moments of a truncated skew-t random variable denoted by $tST(\alpha;v)$ and exposed this calculus through theorems and corollaries.

Theorem 3.2. *If $X \sim tST(\alpha;v)$, then the expectation value of X, is given by*

$$E(X) = \frac{(v+1)^{(\frac{v+1}{2})}v^{(\frac{v+2}{2})}}{2\pi(B-A)}\left\{C_b - C_a - \frac{\left[(v+1)(1+\alpha^2)\right]^{-(\frac{v+2}{2})}}{1-v}\frac{1}{2}(D_a-D_b)\right\}, \tag{3.4}$$

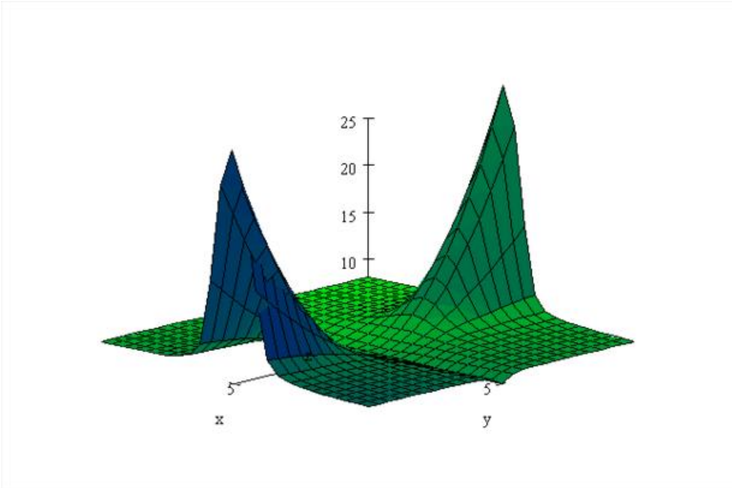

Fig. 2. Graphical representation of Theorem 3.1. Source: own research.

where

$$C_a = \frac{(v+a^2)^{\left(\frac{v+2}{2}\right)}}{1-v}\left[1 - W\left(a\alpha\sqrt{\frac{v+1}{v+a^2}}; v+1; \alpha\right)\right],$$

$$C_b = \frac{(v+b^2)^{\left(\frac{v+2}{2}\right)}}{1-v}\left[1 - W\left(b\alpha\sqrt{\frac{v+1}{v+b^2}}; v+1; \alpha\right)\right],$$

$$D_a = \left(a^2 + \frac{v}{1+\alpha^2}\right)^{-\left(\frac{v+2}{2}\right)}(v+a^2)^2 B\left(\frac{v-2}{2};\frac{1}{2}\right)$$
$$\quad {}_2F_1\left(\frac{v+2}{2};\frac{1}{2};\frac{v+1}{2};\frac{\frac{v}{1+\alpha^2}-v}{a^2+\frac{v}{1+\alpha^2}}\right),$$

$$D_b = \left(b^2 + \frac{v}{1+\alpha^2}\right)^{-\left(\frac{v+2}{2}\right)}(v+b^2)^2 B\left(\frac{v-2}{2};\frac{1}{2}\right)$$
$$\quad {}_2F_1\left(\frac{v+2}{2};\frac{1}{2};\frac{v+1}{2};\frac{\frac{v}{1+\alpha^2}-v}{b^2+\frac{v}{1+\alpha^2}}\right).$$

Proof. The proof is based on generalized hypergeometric series [23]. □

Remark 3.3. *In the next section we will study the convergence of the hypergeometric series in this particular case.*

Theorem 3.4. *If $X \sim tST(\alpha; v)$, then the second moment of X, is*

$$E(X^2) = \frac{(v+1)^{\left(\frac{v+1}{2}\right)}v^{\left(\frac{v+2}{2}\right)}}{2\pi(B-A)}\left(A_bB_b - A_aB_a - (C_1 - C_2)\right), \qquad (3.5)$$

where

$$\begin{aligned}
A_i &= i\frac{(v+i)^{\frac{-v+1}{2}}}{1-v} - \frac{1}{(1-v)v^m}\sum_{k=0}^{m-1}\frac{(-1)^k}{2k+1}\left(\begin{array}{c} m-1 \\ k \end{array}\right)\left(\frac{i}{\sqrt{v+i^2}}\right)^{2k+1}, \\
B_i &= 1 - W\left(i\alpha\sqrt{\frac{v+1}{v+i^2}}; v+1; \alpha\right), \\
C_1 &= \frac{(v+1)^{-(\frac{v+2}{2})}(1+\alpha^2)^{-(\frac{v+2}{2})}}{2(1-v)\left(\frac{-3+v}{2}\right)}(c_1^{**} - c_1^*), \quad \textit{with} \\
c_1^{**} &= (a^2+p)^{\frac{-v+3}{2}}\ {}_2F_1\left(-\frac{3}{2}, \frac{-3+v}{2}; \frac{-3+v+2}{2}; \frac{-(1+\alpha^2)}{(a^2+p)v\alpha^2}\right), \\
c_1^{*} &= (b^2+p)^{\frac{-v+3}{2}}\ {}_2F_1\left(-\frac{3}{2}, \frac{-3+v}{2}; \frac{-3+v+2}{2}; \frac{-(1+\alpha^2)}{(b^2+p)v\alpha^2}\right), \\
C_2 &= \frac{(v+1)^{-(\frac{v+2}{2})}(1+\alpha^2)^{-(\frac{v+2}{2})}}{2(1-v)v^m} \\
&\sum_{k=0}^{m-1}\frac{(-1)^k}{2k+1}\left(\begin{array}{c} m-1 \\ k \end{array}\right)(a^2+p)^{-\frac{v+2}{2}}(v+a^2)^{\frac{3+v}{2}}B\left(-\frac{1}{2}, k+1\right) \\
&{}_2F_1\left(\frac{v+2}{2}, k+1; \frac{v-2k}{2}; \frac{(p-v)}{(a^2+p)}\right) - (b^2+p)^{-\frac{v+2}{2}}(v+b^2)^{\frac{3+v}{2}}B\left(-\frac{1}{2}, k+1\right) \\
&{}_2F_1\left(\frac{v+2}{2}, k+1; \frac{v-2k}{2}; \frac{(p-v)}{(b^2+p)}\right), \quad \textit{with} \ \ m = \frac{v}{2} - 1 \ \ \textit{and} \ \ p = \frac{1}{1+\alpha^2}.
\end{aligned}$$

Proof. The proof is based on generalized hypergeometric series [23]. □

Theorem 3.5. *If $X \sim tST(\alpha; v)$, then the variance of X, is given by*

$$\begin{aligned}
Var(X) \quad &= \quad \left(\frac{(v+1)^{(\frac{v+1}{2})}v^{(\frac{v+2}{2})}}{2\pi(B-A)}\right)^2 \left\{ \frac{[A_bB_b - A_aB_a - (C_1 - C_2)]}{\frac{(v+1)^{(\frac{v+1}{2})}v^{(\frac{v+2}{2})}}{2\pi(B-A)}} \right. \\
&\left. - \left[(C_b - C_a) - \frac{\left[(v+1)(1+\alpha^2)\right]^{-(\frac{v+2}{2})}}{(1-v)}\frac{1}{2}(D_a - D_b)\right]^2 \right\}. \qquad (3.6)
\end{aligned}$$

Proof. $Var(X) = E(X^2) - E^2(X)$. □

Theorem 3.6. *If $X \sim tST(\alpha; v)$, then the n-th moment odd of X, is given by*

$$E(X^n) \quad = \quad \frac{(v+1)^{\left(\frac{v+1}{2}\right)}v^{\left(\frac{v+2}{2}\right)}}{2\pi(B-A)}\left[(D_b - D_a) - W\right], \qquad (3.7)$$

where

$$\begin{aligned}
D_a &= \left[1 - W\left(a\alpha\sqrt{\frac{v+1}{v+a^2}}; v+1; \alpha\right)\right]\left(\frac{n-1}{2}\right)!a^{n-1} \\
&\times \sum_{i=0}^{\frac{n-1}{2}} \frac{(-1)^i\left[-\left(\frac{v+1}{2}\right)\right]!a^{\frac{-i}{2}}\left(v+a^2\right)^{\frac{-v+1+2i}{2}}}{\left(\frac{n-1-2i}{2}\right)!\left(\frac{-v+1+2i}{2}\right)!},
\end{aligned}$$

$$D_b = \left[1 - W\left(b\alpha\sqrt{\frac{v+1}{v+b^2}}; v+1; \alpha\right)\right]\left(\frac{n-1}{2}\right)! b^{n-1}$$
$$\times \sum_{i=0}^{\frac{n-1}{2}} \frac{(-1)^i \left[-\left(\frac{v+1}{2}\right)\right]! b^{\frac{-i}{2}} \left(v+b^2\right)^{\frac{-v+1+2i}{2}}}{\left(\frac{n-1-2i}{2}\right)! \left(\frac{-v+1+2i}{2}\right)!},$$
$$W = \frac{\left(\alpha^2+1\right)^{-\left(\frac{v+2}{2}\right)}}{(v+1)^{\left(\frac{v+2}{2}\right)}}\left(\frac{n-1}{2}\right)!\left[-\left(\frac{v+1}{2}\right)\right]!\left[\sum_{i=0}^{\frac{n-1}{2}}\left(\frac{(-1)^i}{\left(\frac{n-1-2i}{2}\right)!\left(\frac{-v+1+2i}{2}\right)!} H_i\right)\right],$$

where H_i is a function of ${}_2F_1\left(-\left(\frac{3+2i}{2}\right); \frac{2n-i+4}{4}; -\frac{10+3i+2n}{4}; -\frac{\frac{v}{\left(\alpha^2+1\right)}-v}{vc^2}\right)$*, with truncation parameter c.*

Proof. The proof is based on generalized hypergeometric series [23]. □

Theorem 3.7. *If $X \sim tST(\alpha; v)$, then the n-th moment even of X, $n \neq 2$, is given by*

$$E(X^n) \quad = \frac{(v+1)^{\left(\frac{v+1}{2}\right)} v^{\left(\frac{v+2}{2}\right)}}{2\pi(B-A)}\left[(D_b - D_a) - W\right], \tag{3.8}$$

where

$$D_a = \frac{\frac{1}{[-(v+1)]^{\frac{n-2}{2}}}\left(\frac{n-2}{2}\right)!\Gamma\left(\frac{1}{2}\right)\sum_{i=1}^{\frac{n}{2}-1}\frac{(-1)^i(v^2+v+1)^{\frac{n}{2}-1-i}}{\left(\frac{n}{2}-1-i\right)!\Gamma\left(\frac{1}{2}+i+1\right)}\left(\sqrt{a^2+v}\right)^{1+2i}}{\left[1 - W\left(a\alpha\sqrt{\frac{v+1}{v+a^2}}; v+1; \alpha\right)\right]^{-1}},$$
$$D_b = \frac{\frac{1}{[-(v+1)]^{\frac{n-2}{2}}}\left(\frac{n-2}{2}\right)!\Gamma\left(\frac{1}{2}\right)\sum_{i=1}^{\frac{n}{2}-1}\frac{(-1)^i(v^2+v+1)^{\frac{n}{2}-1-i}}{\left(\frac{n}{2}-1-i\right)!\Gamma\left(\frac{1}{2}+i+1\right)}\left(\sqrt{b^2+v}\right)^{1+2i}}{\left[1 - W\left(b\alpha\sqrt{\frac{v+1}{v+b^2}}; v+1; \alpha\right)\right]^{-1}},$$
$$W = \frac{(v+1)^{-\left(\frac{v+2}{2}\right)}(\alpha^2+1)^{-\left(\frac{v+2}{2}\right)}}{[-(v+1)]^{\frac{n}{2}-1}}\left(\frac{n}{2}-1\right)!\Gamma\left(\frac{1}{2}\right)\sum_{i=0}^{\frac{n}{2}-1}\frac{(-1)^i(v^2+v+1)^{\frac{n}{2}-1-i}}{\left(\frac{n}{2}-1-i\right)!\Gamma\left(\frac{1}{2}+i+1\right)}H_i,$$

where H_i is a function of ${}_2F_1\left(-\left(\frac{3+2i}{2}\right); \frac{2n-i+4}{4}; -\frac{10+3i+2n}{4}; -\frac{\frac{v}{\left(\alpha^2+1\right)}-v}{vc^2}\right)$*, with truncation parameter c.*

Proof. The proof is based on generalized hypergeometric series [23]. □

4 Convergence Study

In this section we study the convergence of the hypergeometric series on the moments of a truncated skew-t distribution. The first two moments depend on hypergeometric series, because if this series does not converge, the moments do not exist, so we will study the conditions for which the hypergeometric series converges, in particular, for the parameters that characterize the density and the moments of a truncated skew-t distribution.

In [24] is provided some particular conditions of divergence, when the parameters of hypergeometric series are negative or zero, but in our case the parameters are positive and non-zero, so we use the D'Alembert criterion for convergence. Usually a hypergeometric series is defined as in [20]

$${}_pF_q\left(a_1, a_2, \ldots, a_p; b_1, b_2, \ldots, b_q; z\right) = \sum_{n=0}^{\infty}\frac{(a_1)_n \ldots (a_p)_n}{(b_1)_n \ldots (b_q)_n}\frac{z^n}{n!}, \tag{4.1}$$

where $(a)_n = a(a+1)(a+2)\dots(a+n-1) = \frac{(a_1+n-1)!}{(a_1-1)!}$.

In our case, we study the series ${}_2F_1(a_1, a_2; b_1; z)$, where can be expressed by

$$\begin{aligned} {}_2F_1(a_1, a_2; b_1; z) &= \sum_{n=0}^{\infty} \frac{\frac{(a_1+n-1)!}{(a_1-1)!}\frac{(a_2+n-1)!}{(a_2-1)!}}{\frac{(b_1+n-1)!}{(b_1-1)!}} \frac{z^n}{n!} = \sum_{n=0}^{\infty} \frac{\frac{\Gamma(a_1+n)}{\Gamma(a_1)}\frac{\Gamma(a_2+n)}{\Gamma(a_2)}}{\frac{\Gamma(b_1+n)}{\Gamma(b_1)}} \frac{z^n}{n!} \\ &= \sum_{n=0}^{\infty} \frac{\Gamma(b_1)\Gamma(n)\Gamma(a_1+n)\Gamma(a_2+n)\Gamma(n)}{\Gamma(b_1+n)\Gamma(n)\Gamma(a_1)\Gamma(a_2)\Gamma(n)} \frac{z^n}{n!} \\ &= \sum_{n=0}^{\infty} \frac{B(a_1,n)B(a_2,n)}{B(b_1,n)} \frac{z^n}{n}, \end{aligned} \tag{4.2}$$

where $B(\cdot,\cdot)$ is the Beta function.

Now we present the study of convergence for our particular case, in which the first two moments depend on the functions

$${}_2F_1\left(\frac{v+2}{2}, \frac{1}{2}; \frac{v+1}{2}; \frac{-\alpha^2 v}{b^2+\alpha^2 b^2+v}\right),$$

$${}_2F_1\left(\frac{v+2}{2}, k+1; \frac{v-2k}{2}; \frac{\frac{1}{\alpha^2+1}-v}{b^2+\frac{1}{\alpha^2+1}}\right),$$

$${}_2F_1\left(-\left(\frac{3+2i}{2}\right); \frac{2n-i+4}{4}; -\frac{10+3i+2n}{4}; -\frac{\frac{v}{(\alpha^2+1)}-v}{vc^2}\right),$$

and

$${}_2F_1\left(-\frac{3}{2}, \frac{-3+v}{2}; \frac{-3+v+2}{2}; \frac{-(1+\alpha^2)}{(a^2+p)v\alpha^2}\right).$$

Let ${}_2F_1\left(\frac{v+2}{2}, \frac{1}{2}; \frac{v+1}{2}; \frac{-\alpha^2 v}{b^2+\alpha^2 b^2+v}\right) = \sum_{n=0}^{\infty} \frac{B\left(\frac{v+1}{2},n\right)}{B\left(\frac{v+2}{2},n\right)B\left(\frac{1}{2},n\right)} \frac{\left(\frac{-\alpha^2 v}{b^2+\alpha^2 b^2+v}\right)^n}{n}$, where the general term of the series has the form

$$a_n = \frac{B\left(\frac{v+1}{2},n\right)}{B\left(\frac{v+2}{2},n\right)B\left(\frac{1}{2},n\right)} \frac{\left(\frac{-\alpha^2 v}{b^2+\alpha^2 b^2+v}\right)^n}{n}, \tag{4.3}$$

thus based on D'Alembert criterion and the Equation (4.3), we have that

$$\begin{aligned} \left\{\frac{a_{n+1}}{a_n}\right\} &= \left\{\frac{\frac{\Gamma\left(\frac{v+2}{2}+n+1\right)\Gamma\left(\frac{1}{2}+n+1\right)}{\Gamma\left(\frac{v+1}{2}+n+1\right)}\frac{z^{n+1}}{(n+1)!}}{\frac{\Gamma\left(\frac{v+2}{2}+n\right)\Gamma\left(\frac{1}{2}+n\right)}{\Gamma\left(\frac{v+1}{2}+n\right)}\frac{z^n}{n!}}\right\} = \left\{\frac{\left(\frac{v+2}{2}+n\right)\left(\frac{1}{2}+n\right)}{\left(\frac{v+1}{2}+n\right)}\frac{z}{(n+1)}\right\} \\ &= \left\{\frac{n^2\left(\frac{v+2}{2n}+1\right)\left(\frac{1}{2n}+1\right)}{n^2\left(\frac{v+1}{2n}+1\right)}\frac{z}{\left(1+\frac{1}{n}\right)}\right\} \\ &= \left\{\frac{\left(\frac{v+2}{2n}+1\right)\left(\frac{1}{2n}+1\right)}{\left(\frac{v+1}{2n}+1\right)}\frac{z}{\left(1+\frac{1}{n}\right)}\right\} = z. \end{aligned} \tag{4.4}$$

This means that the convergence of hypergeometric series in this situation depends on z and in cases that the module is less than 1, so ${}_2F_1\left(\frac{v+2}{2}, \frac{1}{2}; \frac{v+1}{2}; \frac{-\alpha^2 v}{b^2+\alpha^2 b^2+v}\right)$ converges if $\left|\frac{-\alpha^2 v}{b^2+\alpha^2 b^2+v}\right| < 1$. Equivalently,

$${}_2F_1\left(\frac{v+2}{2}, k+1; \frac{v-2k}{2}; \frac{\frac{1}{\alpha^2+1}-v}{b^2+\frac{1}{\alpha^2+1}}\right) = \sum_{n=0}^{\infty} \frac{B\left(\frac{v-2k}{2},n\right)}{B\left(\frac{v+2}{2},n\right)B(k+1,n)} \frac{\left(\frac{1-\alpha^2 v-v}{\alpha^2 b^2+b^2+1}\right)^n}{n},$$

using the D'Alembert criterion

$$\begin{aligned}\left\{\frac{a_{n+1}}{a_n}\right\} &= \left\{\frac{\frac{\Gamma\left(\frac{v+2}{2}+n+1\right)\Gamma(k+1+n+1)}{\Gamma\left(\frac{v-2k}{2}+n+1\right)}\frac{z^{n+1}}{(n+1)!}}{\frac{\Gamma\left(\frac{v+2}{2}+n\right)\Gamma(k+1+n)}{\Gamma\left(\frac{v-2k}{2}+n\right)}\frac{z^n}{n!}}\right\} \\ &= \left\{\frac{\frac{\left(\frac{v+2}{2}+n\right)\Gamma\left(\frac{v+2}{2}+n\right)(k+1+n)\Gamma(k+1+n)}{\left(\frac{v-2k}{2}+n\right)\Gamma\left(\frac{v-2k}{2}+n\right)}\frac{z^{n+1}}{(n+1)!}}{\frac{\Gamma\left(\frac{v+2}{2}+n\right)\Gamma(k+1+n)}{\Gamma\left(\frac{v-2k}{2}+n\right)}\frac{z^n}{n!}}\right\} \\ &= \left\{\frac{\left(\frac{v+2}{2}+n\right)(k+1+n)}{\left(\frac{v-2k}{2}+n\right)}\frac{z}{(n+1)}\right\} = z. \end{aligned} \tag{4.5}$$

Therefore,

$${}_2F_1\left(\frac{v+2}{2}, k+1; \frac{v-2k}{2}; \frac{\frac{1}{\alpha^2+1}-v}{b^2+\frac{1}{\alpha^2+1}}\right) \quad \text{converges if} \quad \left|\frac{1-\alpha^2 v - v}{\alpha^2 b^2 + b^2 + 1}\right| < 1,$$

$${}_2F_1\left(-\frac{3}{2}, \frac{-3+v}{2}; \frac{-3+v+2}{2}; \frac{-(1+\alpha^2)}{(a^2+p)v\alpha^2}\right) \quad \text{converges if} \quad \left|\frac{-(1+\alpha^2)}{(a^2+\frac{1}{1+\alpha^2})v\alpha^2}\right| < 1,$$

and equivalently,

$${}_2F_1\left(-\left(\frac{3+2i}{2}\right); \frac{2n-i+4}{4}; -\frac{10+3i+2n}{4}; -\frac{\frac{v}{(\alpha^2+1)}-v}{vc^2}\right) \quad \text{converges if} \quad \left|-\frac{\frac{v}{(\alpha^2+1)}-v}{vc^2}\right| < 1.$$

If the conditions developed above are satisfied then the first two moments of truncated skew-t exist.

Now if $\left|-\frac{\frac{v}{(\alpha^2+1)}-v}{vc^2}\right| < 1$, then ${}_2F_1\left(-\left(\frac{3+2i}{2}\right); \frac{2n-i+4}{4}; -\frac{10+3i+2n}{4}; -\frac{\frac{v}{(\alpha^2+1)}-v}{vc^2}\right)$ converges the n-th moment exists. The proof is based on the D'Alembert criterion for convergence.

So far, we have explained with details the moments of the truncated skew-t distribution and the convergence of hypergeometric series to ensure the existence of such moments, in order to use the results of this distribution with compact support to specify finite traveling times that are considered in the shortest path problem as an application in the next section 5.

5 Application

Considerer the map Fig. 3. We use graphs to model the paths and vertices of different way, which allows visualization and identification of the edges. We wish to determine the shortest path to go from A to B, in the following situations: Let C_i, D_i and E_i, $i = 1, ..., 5$, be the transition times in minutes of their paths, which we assume truncated skew-t distribution for each one of them, that is, $C_i \sim SKTT\left(a_{C_i}, b_{C_i}, \alpha_{C_i}, v_{C_i}\right)$, $D_i \sim SKTT\left(a_{D_i}, b_{D_i}, \alpha_{D_i}, v_{D_i}\right)$ and $E_i \sim SKTT\left(a_{E_i}, b_{E_i}, \alpha_{E_i}, v_{E_i}\right)$, where a_j and b_j are the truncation parameters.

Graph theory allows a wide quantity results for the determination of the optimal paths shorter, such as the establishment of the centroid of the graph, but most results are thought processes deterministic, such as geodesic distance. In this situation would be a map with three paths that quantify the same distance between A and B, but the randomization of the weights allows a redefinition of the geodesic distance using weights of the edges.

Table 1 shows the different parameter values for the truncated skew-t distributions of the transition times C_i, D_i and E_i, and Table 2 illustrates the respective right expectation values.

Fig. 3. Viewing the paths and the vertices.

Table 1. Parameter values for the transition times C_i, D_i and E_i

i	a_{C_i}	b_{C_i}	α_{C_i}	v_{C_i}	a_{D_i}	b_{D_i}	α_{D_i}	v_{D_i}	a_{E_i}	b_{E_i}	α_{E_i}	v_{E_i}
1	5	∞	2	10	12	∞	2	13	4	∞	2	11
2	6	∞	4	10	15	∞	4	13	7	∞	4	11
3	7	∞	5	10	16	∞	5	13	6	∞	5	11
4	6	∞	7	10	9	∞	7	13	9	∞	7	11
5	5	∞	3	10	18	∞	3	13	7	∞	3	11

Table 2. Right expectation values for the transition times

i	1	2	3	4	5
$E(C_i)$	33.5	47.25	63.51	22.21	33.5
$E(D_i)$	17.26	33.83	30.31	9.79	39.35
$E(E_i)$	21.73	62.11	46.22	101.25	62.11

Thus the expected values for the transition times of their paths are:

$$\begin{aligned} E(C_i) &= 33.5 + 47.25 + 63.51 + 22.21 + 33.5 = 199.97 \\ E(D_i) &= 17.26 + 33.83 + 30.31 + 9.79 + 39.35 = 130.54 \\ E(E_i) &= 21.73 + 62.11 + 46.22 + 101.25 + 62.11 = 293.42 \end{aligned}$$

We can observe that D has the least distance between A and B since it has the lowest expected demand, with an average time 130.54 minutes. On the other hand, C requires a average time of 199.97 minutes and E presents the highest average time. Therefore, just is necessary find the equilibrium between time and distance. From graph theory, these results are an approximation to a definition of the shortest path algorithm where the weights of the edges are truncated skew-t random variables.

6 Conclusions

Sometimes, observed data sets are almost exclusively truncated, because of analytical detection limits or spatial and temporal limitations on data collection. Some related proposals and results have appeared in the literature under the concept of the truncated distribution. The present paper considered the moments of a truncated skew-t distribution, giving descriptive information about the distribution. As given in Section 4, in addition to their utility as general descriptive measures, the moments can be employed for solving statistical problems. When we study the functional relationship between the expectation of a truncated skew-t random variable and different values of the asymmetry parameter, α, keeping fixed the degrees of freedom, v, on three levels, changes in the expectation when $\alpha \to \infty$, are null. Future research is intended to build applications in optimal paths from a random viewpoint.

Comments

begindefinition The (differential) entropy of a random vector $\mathbf{Z} \in IR^k$ with probability density function $f_Z(z)$ is defined in [10]

$$H_{\mathbf{Z}} = -E\left[\log\left\{f_Z(\mathbf{Z})\right\}\right] = -\int\limits_{IR^k} \log\left\{f_Z(\mathbf{z})\right\} f_Z(\mathbf{z})\, dz. \tag{6.1}$$

We know that $f_{\alpha,v}(x|a < X \leq b) = \begin{cases} \frac{1}{[F_{\alpha,v}(x)]_a^b} f_{\alpha,v}(x), & \text{if } a < x \leq b \\ 0, & \text{otherwise} \end{cases}$,

where

$$f_{\alpha,v}(x) = \frac{(v+1)^{\left(\frac{v+1}{2}\right)} v^{\left(\frac{v+2}{2}\right)}}{2\pi} (v+x^2)^{-\left(\frac{v+1}{2}\right)} \left(1 - W\left(x\alpha\sqrt{\frac{v+1}{v+x^2}}; v+1; \alpha\right)\right) I_{(a,b]}(x)$$

and

$$[F_{\alpha,v}(x)]_a^b = B - A,$$

where

$$B = \int_{-\infty}^{b} \frac{(v+1)^{\left(\frac{v+1}{2}\right)} v^{\left(\frac{v+2}{2}\right)}}{2\pi} (v+x^2)^{-\left(\frac{v+1}{2}\right)} \left(1 - W\left(x\alpha\sqrt{\frac{v+1}{v+x^2}}; v+1; \alpha\right)\right) dx,$$

and

$$A = \int_{-\infty}^{a} \frac{(v+1)^{\left(\frac{v+1}{2}\right)} v^{\left(\frac{v+2}{2}\right)}}{2\pi} (v+x^2)^{-\left(\frac{v+1}{2}\right)} \left(1 - W\left(x\alpha\sqrt{\frac{v+1}{v+x^2}}; v+1; \alpha\right)\right) dx.$$

Observation: When $a \to -\infty$ and $b \to +\infty$ we have

$$f_X(x; \alpha, v) = 2t_1(x; v) T_1\left(\alpha x \left(\frac{v+1}{v+x^2}\right)^{\frac{1}{2}}; v+1\right).$$

So X stands for a skew-t (ST) distribution with mean 0, skewness parameter α , degrees of freedom ν and density denoted by $ST(\alpha, \nu)$. Now, if $Y = \xi + \varpi X$, where $\xi \in IR$, $\varpi \in IR^+$, we obtain $Y \sim ST\left(\xi, \varpi^2, \alpha, \nu\right)$, and the pdf of Y, given by

$$f_Y(y, \xi, \varpi^2, \alpha, v) = 2t_1(y, \xi, \varpi^2, v) T_1\left(\alpha z \left(\frac{v+1}{v+z^2}\right)^{\frac{1}{2}}; v+1\right), \tag{6.2}$$

where $z = \frac{y-\xi}{\varpi}$. Let $w(z, v) = z\left(\frac{v+1}{v+z^2}\right)^{\frac{1}{2}}$, we have that the pdf in Equation (6.3) is

$$f_Y(y, \xi, \varpi^2, \alpha, v) = 2t_1(y, \xi, \varpi^2, v) T_1\left(\alpha w(z, v); v+1\right).$$

The multivariate case (k dimensional), for this model is presented in [10]. We have $\mathbf{Z}_0 \sim ST_k\left(\mathbf{0}, \mathbf{I}_k, \bar{\eta}, v\right)$, with pdf

$$p_{z_0}\left(\mathbf{z}_0\right) = 2t_k\left(\mathbf{z}_0; v\right) T\left(\sqrt{\frac{v+k}{v+\left\|\mathbf{z}_0\right\|^2}}\bar{\eta}^T\mathbf{z}_0; v+k\right).$$

The entropy for the one-dimensional case is given in [22]. Now in our situation, entropy results from solving

$$\begin{aligned} H\left(X\right) &= E\left(-\log f_X(x;\alpha,v)\right) \qquad (6.3)\\ &= E\left(-\log\left\{\frac{(v+1)^{(\frac{v+1}{2})}v^{(\frac{v+2}{2})}}{2\pi(B-A)}(v+x^2)^{-(\frac{v+1}{2})}\right.\right.\\ &\left.\left.\left[1-W\left(x\alpha\sqrt{\frac{v+1}{v+x^2}};v+1;\alpha\right)\right]I_{(a,b]}(x)\right\}\right)\\ &= E\left(-\log\left(\frac{(v+1)^{(\frac{v+1}{2})}v^{(\frac{v+2}{2})}}{2\pi}\right)+(\frac{v+1}{2})\log(v+x^2)\right.\\ &\left.-\log\left[1-W\left(x\alpha\sqrt{\frac{v+1}{v+x^2}};v+1;\alpha\right)\right]+\log\left(B-A\right)\right)\\ &= -\log\left(\frac{(v+1)^{(\frac{v+1}{2})}v^{(\frac{v+2}{2})}}{2\pi}\right)+(\frac{v+1}{2})E\left(\log(v+x^2)\right)\\ &-E\left(\log\left[1-W\left(x\alpha\sqrt{\frac{v+1}{v+x^2}};v+1;\alpha\right)\right]\right)+\log\left(B-A\right), \qquad (6.4) \end{aligned}$$

where we need to determine

$$E\left(\log\left[1-W\left(x\alpha\sqrt{\frac{v+1}{v+x^2}};v+1;\alpha\right)\right]\right),$$

with $W = \left(x\alpha\sqrt{\frac{v+1}{v+x^2}}\right)^{-v} B(1,v)\ {}_3F_2\left(\frac{v+1}{2};\frac{v}{2};\frac{v+1}{2};\frac{v+1}{2};\frac{v+2}{2};\frac{-v+1}{\left(x\alpha\sqrt{\frac{v+1}{v+x^2}}\right)^2}\right)$.

A hypergeometric series is defined in [20] as

$$ {}_pF_q\left(a_1,a_2,...,a_p;b_1,b_2,...,b_q;z\right) = \sum_{n=0}^{\infty}\frac{\left(a_1\right)_n ... \left(a_p\right)_n}{\left(b_1\right)_n ... \left(b_q\right)_n}\frac{z^n}{n!}, \qquad (6.5)$$

where $\left(a\right)_n = a\left(a+1\right)\left(a+2\right)...\left(a+n-1\right) = \frac{(a_1+n-1)!}{(a_1-1)!}$.

Thus,

$$
{}_3F_2\left(\frac{v+1}{2};\frac{v}{2};\frac{v+1}{2};\frac{v+1}{2};\frac{v+2}{2};\frac{-v+1}{\left(x\alpha\sqrt{\frac{v+1}{v+x^2}}\right)^2}\right) ={}_3F_2\left(\frac{v+1}{2};\frac{v}{2};\frac{v+1}{2};\frac{v+1}{2};\frac{v+2}{2};z\right)
$$

$$
\begin{aligned}
&= \sum_{n=0}^{\infty}\frac{\left(\frac{v+1}{2}\right)_n\left(\frac{v}{2}\right)_n\left(\frac{v+1}{2}\right)_n}{\left(\frac{v+1}{2}\right)_n\cdots\left(\frac{v+2}{2}\right)_n}\frac{z^n}{n!}=\sum_{n=0}^{\infty}\frac{\left(\frac{v}{2}\right)_n\left(\frac{v+1}{2}\right)_n}{\left(\frac{v+2}{2}\right)_n}\frac{z^n}{n!}=\sum_{n=0}^{\infty}\frac{\frac{\left(\frac{v}{2}+n-1\right)!}{\left(\frac{v}{2}-1\right)!}\frac{\left(\frac{v+1}{2}+n-1\right)!}{\left(\frac{v+1}{2}-1\right)!}}{\frac{\left(\frac{v+2}{2}+n-1\right)!}{\left(\frac{v+2}{2}-1\right)!}}\frac{z^n}{n!}\\
&= \sum_{n=0}^{\infty}\frac{\left(\frac{v}{2}\right)!\left(\frac{v}{2}+n-1\right)!\left(\frac{v+1}{2}+n-1\right)!}{\left(\frac{v}{2}+n\right)!\left(\frac{v}{2}-1\right)!\left(\frac{v+1}{2}-1\right)!}\frac{z^n}{n!}=\sum_{n=0}^{\infty}\frac{\Gamma\left(\frac{v}{2}+1\right)\Gamma\left(\frac{v}{2}+n\right)\Gamma\left(\frac{v+1}{2}+n\right)}{\Gamma\left(\frac{v}{2}+n+1\right)\Gamma\left(\frac{v}{2}\right)\Gamma\left(\frac{v+1}{2}\right)n\Gamma\left(n\right)}z^n\\
&= \sum_{n=0}^{\infty}\frac{\Gamma\left(n\right)\Gamma\left(n\right)}{B\left(\frac{v}{2}+1;n\right)B\left(\frac{v}{2};n\right)B\left(\frac{v+1}{2};n\right)n}z^n, \qquad (6.6)
\end{aligned}
$$

where $\Gamma(\cdot)$ is the Gamma function and $B(\cdot)$ is the Beta function.

Now, $E\left(\log\left(1-W\left(x\alpha\sqrt{\frac{v+1}{v+x^2}};v+1;\alpha\right)\right)\right)$

$$
\begin{aligned}
&= E\left(\log\left\{1-\left[\left(x\alpha\sqrt{\frac{v+1}{v+x^2}}\right)^{-v}B(1,v)\right]\right.\right.\\
&\qquad \left.\left.{}_3F_2\left(\frac{v+1}{2};\frac{v}{2};\frac{v+1}{2};\frac{v+1}{2};\frac{v+2}{2};\frac{-v+1}{\left(x\alpha\sqrt{\frac{v+1}{v+x^2}}\right)^2}\right)\right\}\right)\\
&= E\left(\log\left[1-\left(x\alpha\sqrt{\frac{v+1}{v+x^2}}\right)^{-v}B(1,v)\sum_{n=0}^{\infty}\frac{\Gamma\left(n\right)\Gamma\left(n\right)}{B\left(\frac{v}{2}+1;n\right)B\left(\frac{v}{2};n\right)B\left(\frac{v+1}{2};n\right)n}z^n\right]\right)\\
&= E\left(\log\left[1-\left(\frac{1}{x\alpha}\right)^{v}\left(\frac{v+x^2}{v+1}\right)^{\frac{v}{2}}\right.\right.\\
&\qquad \left.\left.\sum_{n=0}^{\infty}\frac{B(1,v)\Gamma\left(n\right)\Gamma\left(n\right)}{B\left(\frac{v}{2}+1;n\right)B\left(\frac{v}{2};n\right)B\left(\frac{v+1}{2};n\right)n}\left(\frac{\left(v+x^2\right)\left(1-v\right)}{x^2\alpha^2\left(v+1\right)}\right)^n\right]\right)\\
&= E\left(\log\left[1-\sum_{n=0}^{\infty}\frac{B(1,v)\Gamma\left(n\right)\Gamma\left(n\right)}{B\left(\frac{v}{2}+1;n\right)B\left(\frac{v}{2};n\right)B\left(\frac{v+1}{2};n\right)n}\frac{\left(v+x^2\right)^{\frac{v}{2}+n}\left(1-v\right)^n}{x^{2n+v}\alpha^{2n+v}\left(v+1\right)^{\frac{v}{2}+n}}\right]\right), \qquad (6.7)
\end{aligned}
$$

which is still an open problem.

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Competing Interests

Authors have declared that no competing interests exist.

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